UNIVERSITY OF WESTERN MACEDONIA

DEPARTMENT OF MECHANICAL ENGINEERING

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Aeronautical Engineer

DEVELOPMENT AND ANALYSIS OF ADVANCED ADAPTIVE STATISTICAL PROCESS CONTROL CHARTS FOR THE JOINT MONITORING OF VARIABLES' CENTRAL TENDENCY AND DISPERSION

Ph.D DISSERTATION

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Ph.D DISSERTATION

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«Η έγκριση της παρούσας Διδακτορικής Διατριβής από το Τμήμα Μηχανολόγων Μηχανικών του Πανεπιστημίου Δυτικής Μακεδονίας δεν υποδηλώνει αποδοχή των γνωμών του συγγραφέως» (Ν. 5343/1932, άρθρο 202, παρ. 2)

“*What a man can be, he must be*.”

Abraham Maslow

ABSTRACT

Fully adaptive control charts are efficient Statistical Process Control (*SPC*) means to monitor a quality characteristic affecting the outcome of a manufacturing process. Usually, the performance of these adaptive charts is investigated in processes characterized by a single assignable cause mechanism. However, this assumption is frequently far from reality because a process shift to the out-of-control condition can be the consequence of several assignable causes, which can occur at the same time, or independently, and may affect the process mean, the standard deviation of the process, or both.

Furthermore, the logical link between quality control and equipment maintenance and the improvement of the process performance in case of the incorporation of proactive actions to process monitoring techniques necessitated the study of their interaction under a multiplicity of assignable causes.

Another complicated issue is the simultaneous monitoring of multiple correlated quality characteristics, which is, undoubtedly, crucial in today's process applications. The independent monitoring of these multiple quality characteristics, especially in the presence of high correlation between them, may lead to erroneous monitoring policies.

In this thesis, Variable-Parameter (*VP*) Shewhart control schemes, monitoring the location and scale of processes in presence of multiple assignable causes are presented. All the proposed control schemes are both economically and statistically optimized. Furthermore, for each of the proposed control schemes a Markov chain that models the occurrence of several assignable causes leading to progressive process deterioration, and calling for different corrective actions, is developed.

The motivation and contribution of this thesis precede a detailed literature review. In the literature review, statistically optimized control schemes utilized for the joint monitoring of location and scale of processes are, firstly, presented. Then, some models both economically and statistically optimized for monitoring the mean and the dispersion of a process are pointed out. Partially adaptive and fully adaptive control charts, that have a better economic and statistical performance compared to their respective static ones, are also presented. Furthermore, studies on control charts for

monitoring processes subject to a multiplicity of assignable causes and a detailed review on integrated maintenance and quality control schemes are presented. Finally, multivariate control charts for monitoring processes subject to multiple assignable causes affecting both location and scale can be found in the literature review.

The problem setting and the assumptions of the proposed models both for univariate and multivariate processes are also presented in detail. Furthermore, every statistical measure utilized to define the statistical performance of the proposed control schemes is presented. The extension of the models to the realistic cases of imperfect process restoration and downward affection of the process mean increase significantly the applicability of the control schemes, introduced in the thesis.

The first proposed *SPC* model is a new economic-statistically optimized *VP* control scheme for the optimization of a process operation where two assignable causes may occur, one affecting the mean and the other the standard deviation of the process. Therefore, it is possible for the process to operate in statistical control, when none of the two assignable causes has occurred, or under the effect of one, or both the assignable causes. The superiority of the proposed model is estimated by comparing its expected total quality-related costs vs. the economic outcome of the respective static and partially adaptive control schemes, for a benchmark of numerical examples. The numerical investigation indicates that the economic improvement of the proposed model may be significant.

Moreover, the economic-statistical design of a *VP* control chart monitoring the process mean in presence of multiple assignable causes affecting the location of the process is presented. A benchmark of examples has been generated to compare the performance of the *VP* control chart with other less-adaptive control charts and the Fixed-Parameter (*FP*) control chart. The obtained results reveal the economic superiority of the *VP* control chart.

The problem of the possible occurrence of multiple assignable causes that may affect both the location and scale of the monitored process is investigated. Subsequently, the economic-statistical design of a *VP* Shewhart control scheme for monitoring processes where multiple assignable causes, affecting both the mean and the dispersion of the process, is presented. The assignable causes may lead to progressive process deterioration and their simultaneous occurrence and the different corrective action for each assignable cause makes the proposed model more realistic. An extended numerical investigation is utilized to demonstrate the economic and statistical superiority of the proposed model against simpler approaches. An example from aviation industry illustrates the application of the model.

Furthermore, a new *VP* Shewhart control scheme is presented for the economic-statistical optimization in cases where apart from multiple independent assignable causes, affecting both the mean and dispersion, failures may also occur. Each time the control scheme signals an alarm, preventive maintenance (*PM*) actions are initiated which are obviously preferable to corrective maintenance (*CM*) actions, required after a failure. The realistic assumption of imperfect *PM* actions has been considered. The optimal design parameters of the scheme are selected through a bi-objective optimization problem formulated by the long-run average cost per time unit minimization, and the long-run expected availability maximization, subject to statistical constraints. An extended numerical investigation is utilized to demonstrate the superiority of the proposed model against simpler control schemes.

A new fully adaptive multivariate statistical process control (m-*SPC*) scheme for monitoring processes where multiple assignable causes may occur is studied. The assignable causes are independent and affect both the mean vector and the covariance matrix, which are monitored by a T^2 control chart and a multivariate Shewhart control chart based on differential entropy, respectively. A real case example is employed to illustrate the operation of the proposed model and measure its economic and statistical performance for the specific example.

Finally, the basic conclusions of this thesis, which are presented in Chapter 10, can be summarized in the following:

- The development of the proposed, easy-to-use, monitoring tools allow the simultaneous monitoring of both the location and scale of processes under a multiplicity of assignable causes. Moreover, the effective monitoring of processes where, except for multiple quality shifts, failures are also possible to occur, is now feasible. Finally, the development of a control scheme for the simultaneous monitoring of multiple correlated quality characteristics allow the monitoring of multivariate processes when both the process location and variability are affected by multiple shifts.

- All the proposed fully adaptive control schemes have a better economic and statistical performance compared to the respective, less-adaptive ones. Subsequently, the proposed schemes lead to significant cost savings and also enhance the confidence of practitioners to the control procedure.

- The incorrect consideration of a single instead of multiple assignable causes imposes a significant cost to the process. This conclusion necessitates the application of the proposed control schemes in modern processes where the assumption of only one possible quality shift is, in most cases, far from reality.

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This thesis is the outcome of a both enjoyable and painful journey, fueled by passion, self-discipline and hard work. However, this amazing experience would not have been possible without the consistent support and guidance I received from many people, who I would like to thank.

According to Xenophon's Memorabilia: «*μὴ γίγνεσθαι σπουδαίους ἄνευ διδασκάλων ίκανών*» [You cannot achieve excellence without competent teachers]. So, first and foremost, I would like to express my deepest gratitude and admiration to my advisor Assistant Professor George Nenes, who has been a competent teacher and a tremendous mentor for me. His guidance, support, advice and friendship all these years have been priceless. It has been a great honor to be his first Ph.D. student.

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1. INTRODUCTION

1.1 Overview

Control charts are widely adopted *SPC* tools for monitoring the quality of processes both in the industrial and service sectors, because, on-line *SPC* is probably the most efficient method to monitor and control the quality of a process when process disturbances are not self-announced. In general, samples are taken at specified intervals and by plotting a statistic on the control chart, a practitioner can estimate whether the process operates in-control (*IC*) or out-of-control (*OOC*).

The implementation of a control chart requires the careful selection of its design parameters. For the well-known *FP* Shewhart chart, the sample size n, the sampling interval h and the width of control limit coefficients k should be selected. The implementation of an adaptive control chart allows each design parameter to vary at different levels depending on the position of the last point(s) plotted on the chart. Usually, adaptive Shewhart control charts use two values for each design parameter, i.e., two levels are allowed for sample sizes (n_1, n_2) and/or sampling intervals (h_1, h_2) and/or warning limit coefficients (w_1, w_2) and/or control limit coefficients (k_1, k_2); for further details see Costa (1994, 1997, 1999b). Depending on which parameters are allowed to vary, the adaptive control charts are classified in:

- Variable Parameter (*VP*) control charts, if $(n_1 \neq n_2)$, $(h_1 \neq h_2)$ and $(k_1 \neq k_2)$.
- Variable Sample Size and Sampling Interval (*VSSI*) control charts, if $(n_1 \neq n_2)$, $(h_1 \neq h_2)$ and $(k_1 = k_2)$.
- Variable Sample Size (*VSS*) control charts, if $(n_1 \neq n_2)$, $(h_1 = h_2)$ and $(k_1 = k_2)$.
- Variable Sampling Interval (*VSI*) control charts, if $(n_1 = n_2)$, $(h_1 \neq h_2)$ and $(k_1 = k_2)$.

The warning limit coefficients (w_1, w_2) may be adaptive or not in all aforementioned cases. Other adaptive control charts, less known to practitioners, are not considered in this thesis.

Apart from the statistical design of a control chart, where the selection of the design parameters aims at optimizing its statistical performance under a constraint usually regarding the number of false alarms, it is evident that the economic impact of on-line quality control monitoring should be considered when a control chart is designed. The economic design of a control chart allows the chart design parameters to be selected so as to minimize the expected quality control cost per time unit.

Typically, multiple assignable causes can have a critical effect on quality characteristics, the variability of which is influenced by several process parameters, activities and their interactions. An assignable cause shifting one of the parameters away from its target value not only can induce a shift in the quality characteristic, but can also increase the occurrence rate of any future assignable cause shifting the same or other process parameters from their nominal conditions, thus, further aggravating the process operating state.

For example, manufacturing of plastic discrete parts by means of injection molding is a process where shrinkage should be maintained under control. Shrinkage usually depends on mold temperature, screw speed and their interaction. The occurrence of an assignable cause affecting temperature results in a shrinkage shift, which usually increases with temperature. Furthermore, the viscosity of pelletized raw material also changes with temperature, thus affecting the polymer flow throughout the injection unit. This change of physical condition for the raw material can influence the screw speed selection and lead to a larger shrinkage shift away from its target value.

In the beverage industry, the plastic (made from Polyethylene terephthalate (*PET*)) bottle blowing is performed at high temperatures by a machine where the set point temperature is achieved by means of a series of high wattage lamps. The monitored critical-to-quality characteristic is the *PET* bottle wall thickness measured at specific points of a bottle. If one or more lamps do not work correctly or fail, the wall thickness is affected: clearly, the larger the number of lamps showing anomalies, the worse the state of process deterioration.

Similarly, in the production of high precision profiles of mechanical parts, errors with respect to profiles specified in the engineering drawing should be maintained under control. Several assignable causes can lead to an error: for example, shifts in the target position of any controlled axis of the computer numerical control (*CNC*) machine, unexpected bearing wear, tool failures and fixture positioning errors. If a fixture position error occurs, not only the actual profile shifts away from the specified profile, but also process deterioration to a worse operating state can be induced by the increased probability of having a sudden tool failure due to the fixture position error.

In general, the proposed approaches can be applied to the well-known class of Markov processes modeled by reliability theory (see Rausand and Høyland, 2004). On-line process monitoring is implemented on a distribution parameter of a critical quality characteristic of products manufactured in processes characterized by a set of relevant functioning states. These states, characterizing the process, can be ordered from the best functioning, which in the *SPC* context is identified as the in-control operating condition, to the worst functioning state, that is, the one corresponding to the largest expected shift of the distribution parameter due to the presence of multiple assignable causes. Although the first quality control charts monitored the process mean of a specific quality characteristic, it is often desirable for a control scheme to monitor the standard deviation or to monitor simultaneously the mean and the standard deviation in a process where multiple assignable causes may affect both the location and scale of the quality characteristics.

In addition to quality control, a main objective of contemporary industrial processes is to achieve ongoing high reliability along with low maintenance costs. Several general types of maintenance philosophies have been developed in literature to reduce unexpected breakdowns which have a twofold negative effect: they reduce the efficiency of the process and they incur high restoration costs. The most sophisticated maintenance policy is known as condition-based maintenance (*CBM*), where *PM* actions are carried out based on condition monitoring techniques. *CBM* is normally preferable to *CM* actions, required after a breakdown.

As it was explicitly stated above, quality control is also an important component of a production process, as it is common in industry applications that deterioration to less desirable operating states precedes an unplanned failure. It is apparent that an intuitive relationship between *CBM* and *SPC* exists in real applications. Both of these interrelated scientific fields are based on condition monitoring techniques and their effective integration can definitely improve the process performance.

Finally, in recent years, the rapid development of technology has led to even more sophisticated processes, where quality should be controlled through the simultaneous monitoring of several, usually correlated, quality characteristics. The growth of data acquisition methods has also been an important factor for an increase in research interest in the field of Multivariate Statistical Process Control and, especially, Multivariate Control Charts.

1.2 Motivation

The on-line monitoring of either only the location or only the scale of a process, as it is commonly found in literature and industry, is not an overall optimized approach. Moreover, the presumption of a single assignable cause that may affect the quality characteristic is a simplified hypothesis, which is rarely found in practical applications.

There is a need for advanced mathematical models to deal with the complicated problem of simultaneous monitoring of location and scale of processes being affected by multiple assignable causes. The comprehensive study of sophisticated models aims at producing easy-to-use monitoring tools for practitioners, despite the complexity of the stochastic models, in a wide variety of industries and services.

Furthermore, the economic-statistical design of the aforementioned models can lead to the reduction of the costs associated with the quality operations, which according to Montgomery (2009) are quite large and may even be as high as 35-40% of total operations.

Moreover, the decision-making on quality monitoring and equipment maintenance separately results in suboptimal solutions. Consequently, another existing challenge in today's industry is the integration of quality and maintenance in case failures and multiple quality shifts are possible to occur.

Finally, in cases where the assumption that the quality of the monitored process can be adequately characterized by a single quality characteristic cannot be properly justified or is untrue, multivariate control schemes should be developed. It is apparent that the use of multiple univariate control charts is a simplistic and not effective solution. The fact that the majority of processes in real applications are fully characterized by multiple correlated quality characteristics necessitates the study of multivariate control schemes for processes subject to a multiplicity of assignable causes where location and scale are affected.

1.3 Contribution

The novel contribution of this thesis lies to the development of advanced models for the joint economic-statistical optimization of fully adaptive control schemes for monitoring the location and scale of processes subject to multiple, independent assignable causes. The fact that the assignable causes may affect not only the mean but also the variability of the quality characteristics necessitates the study of general control schemes, which have not been studied before, and are applicable to a wide variety of processes.

Additionally, the integration of on-line quality control and maintenance in complex environments, subject to a multiplicity of assignable causes and failures, results in a state-of-the-art monitoring tool optimized with respect to the average cost per time unit and the equipment availability.

Finally, the joint monitoring of multiple correlated quality characteristics in the presence of multiple quality shifts that affect both the process location and scale deals with a difficult, yet very realistic problem, especially in modern applications.

1.4 Structure of Dissertation

The structure of the dissertation is as follows. In Chapter 2, a detailed literature review on control charts is presented. Chapter 3 provides the general assumptions of the monitored processes, the model formulation and the employed optimization method. Moreover, the utilized statistical measures and possible extensions of the proposed control schemes are given. In Chapter 4, a *VP* control scheme where two assignable causes may occur, one affecting the process mean and one the process variability, is presented.

Chapter 5 provides the study of a control scheme utilized for monitoring processes, where multiple assignable causes affecting only the process location, may occur. Chapter 6 describes, in detail, the employed approach for the complicated problem of computing the transition probabilities and the *OOC* operation cost, in cases where multiple assignable causes, affecting both location and scale, may occur. In Chapter 7, a control scheme for univariate processes subject to multiple quality shifts affecting the process mean and dispersion is studied.

Chapter 8 develops an integrated maintenance and quality control scheme, where apart from multiple quality shifts, failures that cease the process operation may also occur. In Chapter 9, the study of a fully adaptive control scheme for multivariate processes in the presence of a multiplicity of quality shifts affecting the process location and scale is provided. Finally, Chapter 10 concludes the dissertation along with some suggestions on future research.

2. LITERATURE REVIEW

During the last decades several scientists have proposed various statistical quality control charts for monitoring a wide variety of processes. The control charts are utilized in order to dictate changes in the process inputs based on the monitoring of the process outputs so as to achieve operation of the process in the *IC* state (Montgomery, 2009). The first quality control chart was proposed by Shewhart (1931), who has inspired many scientists until now. In that pioneering work, a control chart is employed to monitor the process mean of a specific quality characteristic. However, many control charts have been developed ever since. A thorough review on control charts is developed in this chapter. Nevertheless, due to the large amount of studies on the specific scientific field and for readability reasons, the literature review is divided into categories as described below.

Although the first quality control charts that appeared in literature were designed to monitor the process mean, it is often desirable to monitor the standard deviation or to monitor simultaneously the mean and the standard deviation of a specific quality characteristic. Consequently, several control schemes for the joint monitoring of both parameters have been developed.

Moreover, the benefit of reducing the quality-related costs achieved by the economic design of control charts urged many researchers to utilize the minimization of the expected quality cost per time unit as a main objective of any control chart.

Furthermore, it has soon become evident by many researchers that allowing the values of the design parameters to vary according to the results of the sampling procedure, can improve significantly the economic and statistical performance of a control chart. To this effect, many scientists have developed control charts with adaptive design parameters.

The simplified assumption of a single assignable cause mechanism is unlikely to occur in several manufacturing scenarios due to the usual complexity of production processes: thus, a control chart can have a poor economic performance if the assignable cause behind the shift is different from the one anticipated at the design stage of the chart. To overcome this problem, a stream of research has considered the design of control charts for processes with multiple assignable causes.

Additionally, the interrelationship between maintenance and quality control urged the development of integrated schemes to deal with the quality deterioration and failures of the equipment.

Finally, the fact that the quality of many industrial applications is fully characterized by multiple correlated quality characteristics led to an increasing research interest for multivariate quality control.

In the following sections, a detailed review on the literature on the aforementioned categories of control charts is presented.

2.1 Joint Monitoring of Process Location and Scale

Rahim (1989), Saniga (1991) and Gan (1989, 1997) proposed combined schemes for joint monitoring of the mean and the standard deviation of a process. Moreover, Gan (1995) investigated the use of Exponentially Weighted Moving Average (*EWMA*) control charts for detecting shifts affecting both the process mean and/or the standard deviation, whereas, Chen *et al*. (2001), Khoo and Yap (2005) and Costa and Rahim (2006a) proposed a single *EWMA* control scheme for the same purpose.

Costa and Rahim (2004a) proposed a statistically designed $\overline{X}$ and R scheme where sampling is performed in two stages. During the first stage, one item of the sample is inspected and, depending on the result, the sampling is interrupted if the process is found to be in control; otherwise, it goes on to the second stage, where the remaining items of the sample are inspected. He and Grigoryan (2006) utilized a genetic algorithm to statistically optimize a joint double-sampling $\overline{X}$ and s chart.

Costa and Rahim (2004b) proposed the statistical design of an *EWMA* control chart that is based on the non-central chi-square statistic, for detecting assignable cause(s) that change the process mean and/or increase the variability.

Synthetic control charts based on the non-central chi-square statistic for monitoring both the mean and the standard deviation of a process were investigated by Costa and Rahim (2006b) and Costa *et al*. (2009).

2.2 Economic-Statistical Design

All the previous approaches are statistically optimized, namely, the design parameters of the chart(s) are selected so as to satisfy specific statistical measures of performance. Nevertheless, it is apparent that the estimation of the optimum design parameters of a control chart plays a substantial role for the economic impact of monitoring a process.

When the charts need to be implemented in long run processes, researchers usually refer to the models proposed by Duncan (1956) and Lorenzen and Vance (1986). Another stream of research considered Taguchi's loss function, (Taguchi, Elsayed, and Hsiang, 1989; Spiring and Yeung, 1998) as the reference cost model. Some issues about the economic design of a control chart can be found in a recent review by Celano (2011).

After Duncan's (1956) pioneering work, where the first fully economic model was presented, a stream of research implemented the cost-minimizing criterion for the design of control charts that monitor only the mean of a process (see for example Bather, 1963, Knappenberger and Grandage, 1969 and Gibra, 1971).

On the other hand, only few scientists have developed control charts for monitoring process dispersion by the use of economic criterions. In particular, Trovato *et al*. (2011) and Castagliola *et al*. (2011) proposed economically optimized s (standard deviation) control charts.

Economically optimized control charts for joint monitoring of the process mean and variance have been proposed by Saniga (1977, 1989), Saniga and Montgomery (1981), Jones and Case (1981) and Mcwilliams *et al*. (2001). Furthermore, Costa (1993) proposed the economic design of a control chart where the process is subject to two independent assignable causes, one affecting the mean and the other the variance of the process.

In the same direction, Rahim and Costa (2000) and Costa and Rahim (2000) presented economically optimized control schemes when two independent assignable causes may occur, whose occurrences follow Weibull distributions with increasing failure rates. Another paper that discussed the economic and economic-statistical design of an *EWMA* control chart for joint monitoring of process mean and variance was proposed by Serel and Moskowitz (2008).

Finally, Lu *et al*. (2013) investigated the economic-statistical design of a single *MaxEWMA* control chart, for monitoring both the mean and the variability of a process, by employing two different *EWMA* statistics, one for the mean and one for the variance, but utilizing the maximum of them as the only monitoring statistic.

2.3 Adaptive Control Charts

The common characteristic of the above approaches is that they all assume fixed design parameters. Reynolds *et al*. (1988) were the first to introduce an adaptive control chart by allowing the sampling interval to take either a small or a larger value, depending on the observation of the previous sample. Several *VSI* control charts have been proposed ever since. Das *et al*. (1997), Cui and Reynolds (1988) and Bai and Lee (1998) presented *VSI* control charts from an economic perspective.

Adaptive control charts, where the sample size, instead of the sampling interval, is allowed to vary (*VSS*), were firstly introduced and economically optimized by Prabhu *et al*. (1993) and Park and Reynolds (1994). Moreover, the economic design of *VSSI* control charts has also been investigated by Das and Jain (1997) and Park and Reynolds (1999). Celano *et al*. (2006) proposed statistically designed *VSSI* control charts combined with run rules.

The obvious conclusion of the aforementioned approaches, namely that adaptive control charts have a better economic performance compared to static control charts, led to the study of fully adaptive control charts, where all design parameters are allowed to vary depending on the sampling outcome.

Exhaustive information about the economic design of adaptive control charts monitoring the process mean can be found in Prabhu *et al*. (1997) and De Magalhães *et al*. (2002). Moreover, De Magalhães *et al*. (2001), Costa and Rahim (2001) and

Nenes (2011) proposed fully adaptive control charts that monitor the mean of a process from an economic point of view. A very interesting adaptive model was proposed by Celano *et al*. (2008). In that paper an economically designed adaptive Bayesian chart is proposed, but the assignable cause is assumed to affect only the dispersion of the process.

The common conclusion of the above researches, i.e., the economic and statistical superiority of the adaptive control charts compared to their respective static ones, has attracted the interest of scientists for adaptive control charts that monitor both the mean and the standard deviation of a process. However, due to their increased complexity, adaptive control charts for joint monitoring of process location and scale have been studied by few scientists.

In particular, Costa (1999a) developed a *VSSI*, joint $\overline{X}$ and *R* control chart, optimized by the use of statistical performance criteria while Costa (1998) developed a statistically optimized *VP* control chart for joint monitoring of the process mean and variance. Reynolds and Stoumbos (2001) and Stoumbos and Reynolds (2005) investigated combinations of *VSI EWMA-VSI* $\overline{X}$ control schemes for the problem of monitoring simultaneously the mean and the variance of a process. In a different context, Ohta *et al*. (2002) presented an economic model where there are two assignable causes (affecting the mean and the variance) but the failure mechanism of each cause is governed by a Weibull distribution.

Furthermore, De Magalhães and Moura Neto (2005) discussed the economic optimization of a fully adaptive control chart that monitors jointly the process mean and the variance, which are affected by a single assignable cause mechanism. De Magalhães *et al*. (2006) proposed an adaptive, statistically optimized control chart for monitoring a process subject to two independent assignable causes that affect the process mean and/or the variance. Costa and De Magalhães (2007) extended the work of Costa and Rahim (2004b) and developed a statistically designed *VP* control chart that is also based on the non-central chi-square statistic, for detecting assignable cause(s) that affect the process mean and/or the variability.

De Magalhães *et al*. (2009) proposed a Markovian model for the design of a hierarchy of adaptive $\overline{X}$ control charts. They showed that it is sometimes equally effective to use a chart with fewer varying parameters, depending on the size of the process shift, and yet achieve good statistical performance. Thus, they did not allow for the more general problem setting, where the two assignable causes are independent. Additionally, in their model, the design parameters (n, h, w, k) are not allowed to take every possible values, since the optimization procedure necessitates specific rules concerning the relationship between the relaxed and tightened design parameters. The design parameters are not adaptive but time-varying in a way to assure that the failure probabilities, the sample size per time unit and the power of the charts remain constant at each sampling interval.

Wu *et al*. (2007) proposed a *VSSI* Cumulative Sum (*CUSUM*) control scheme for joint monitoring of mean and variance. Moreover, Tasias and Nenes (2012) developed an economically optimized, fully adaptive, Shewhart control scheme for processes subject to disturbances that may affect independently the process mean and the variance. Finally, Nenes and Panagiotidou (2013) developed an economically optimized, fully adaptive, Bayesian control chart for processes subject to disturbances that may affect independently the process mean and the variance.

2.4 Multiple Assignable Causes

All cited references assume either one assignable cause, the occurrence of which may affect the mean and/or the standard deviation of the process, or two independent assignable causes that affect the mean and the process variance.

Duncan (1971) extended the economic design of the Shewhart $\overline{X}$ control chart to the multiple assignable cause scenario; similarly, Chiu (1976) investigated the economic design of the *np* charts; both these models assume the same reaction (correct detection and perfect restoration) to the occurrence of any assignable cause. This assumption has been debated by Tagaras and Lee (1988) who proposed an economic model for control charts with different control limits for different assignable causes; two levels of corrective actions are considered depending on the adjustment required by the process. More recently, the economic design of a multi-attribute control chart for not overlapping multiple causes has been discussed in Jolayemi (2000). Chen and Yang (2002) considered Weibull in-control times for the process and investigated the effect of the increasing failure rate on the performance of the multiplicity-cause model vs. the single-cause model. The economic design of *VSI* control charts with multiple assignable causes has been investigated in Yu and Hou (2006).

An important criticism to the economic design of a control chart is supported by the consideration that a set of economically optimal design parameters can lead to a poor statistical performance (Woodall, 1986, 1987). For this reason, the cost optimization problem is often constrained by a lower bound for the expected number of samples to be taken between two successive false alarms (i.e., the in-control Average Run Length of the chart). The economic-statistical optimization of $\overline{X}$ Shewhart control charts with multiple assignable causes has been investigated in Asadzadeh and Koshalhan (2009) and Yu *et al*. (2010): statistical constraints were set on the *Type I* and *II* errors to limit the space of feasible solutions of the economic design problem.

With the exception of Tagaras and Lee (1988), all cited references assume either that the occurrence of some assignable cause blocks the possible occurrence of another cause or, alternatively, call for the same corrective action, regardless of the occurring special cause and its effect on the process. The statistical performance of *VSSI* control charts vs. other control charts when the occurrence of an assignable cause does not necessarily prevent the occurrence of another assignable cause has been investigated by Lee *et al*. (2007). The proposed approach is different from a recently published research, (Celano *et al*., 2011), where *VP* control charts have been economically designed for processes subject to a single assignable cause and corrective action, but random shift size.

It should be mentioned that due to their increased complexity, control charts for simultaneous monitoring of the mean and the standard deviation of a process, where multiple assignable causes may occur, are firstly studied in this thesis and are previously published in Tasias and Nenes (2016a).

2.5 Integrated Maintenance and Quality Control Schemes

The first condition-based preventive maintenance models were proposed by Derman (1962, 1963). The first joint optimization of *CBM* and *SPC* can be found in

Paté-Cornell *et al*. (1987) and Tagaras (1988). Moreover, Rahim and Banerjee (1993) presented an age-dependent preventive policy to tackle the problem of equipment failure for processes subject to deterioration. Ben-Daya and Duffuaa (1995), Duffuaa and Ben-Daya (1995), Linderman *et al*. (2005) and Ivy and Nembhard (2005) also recognized the strong link between maintenance and quality.

An economically optimized $\overline{X}$ control chart for the integration of *SPC* and maintenance management was studied by Cassady *et al*. (2000). Chiu and Huang (1996), Ben-Daya and Rahim (2000) and Lee and Rahim (2001) also presented the economic design of a $\overline{X}$ control chart in order to combine preventive maintenance policies with quality control. Moreover, Yeung *et al*. (2007) formulated a Markovian process and achieved economic optimization by utilizing a control chart for the process mean in conjunction with age-replacement maintenance policy. Zhou and Zhu (2008) analyzed the cost of an integration of a control chart and maintenance actions by assuming an increasing hazard rate of the failure mechanism.

Some researchers proposed adaptive control charts to integrate quality and maintenance actions. Kuo (2006) utilized a Partially Observable Markov decision process to determine the value of the sample size and sampling interval based on the current state of the process. Wu and Wang (2011) and Ho and Quinino (2012) studied the economic design of *VSI* control charts for the integration of on-line statistical control and maintenance and concluded that the proposed models achieved a significant economic improvement against the respective static ones. A *VP* $\overline{X}$ control chart was employed by Panagiotidou and Nenes (2009) in order to monitor processes where both quality shifts and failures are possible to occur.

The majority of the aforementioned papers consider control charts that monitor only the process mean of the quality characteristic. There are few studies in the literature that monitor the process dispersion through the formulation of a quality and maintenance model. However, quality shifts that affect both the mean and the variability of a process are commonly found in practical applications. Chiu and Huang (1995) presented the economic design of joint $\overline{X}$ and S^2 control charts for preventive maintenance. The economic-statistical design of control charts that monitor jointly the mean and variability of the process was studied by Caballero Morales (2013) who considered both static and *VSI* control charts for the integration of *PM* with general failure rates. Yin *et al*. (2015) also pointed out the need of joint monitoring of mean and variance in multi-device integrated *SPC* and maintenance models.

In existing literature of integration of *SPC* and maintenance, only a few scientists have considered multiple out-of-control states. Xiang (2013) utilized a discrete-time Markov chain to monitor a process with multiple unobservable out-of-control states which result in the deterioration of the process. Panagiotidou and Tagaras (2010) presented a model for monitoring processes characterized by multiple operational states. However, to the best of our knowledge, the complicated scenario of multiple quality shifts affecting not only the mean but, also, the variability of the process has been studied by Tasias and Nenes (2016b) for the first time.

In the existing literature of *SPC* and *CBM* both the objective function of cost minimization (see, for example, Chiu and Huang, 1995, 1996, Ben-Daya and Rahim, 2000, Panagiotidou and Nenes, 2009 and Ho and Quinino, 2012) and the objective function of availability maximization (see, for example, Barlow and Hunter, 1960 and Jiang *et al*., 2012) have been employed. Nevertheless, the trade-off between these competing objectives has not been explored in the literature.

2.6 Multivariate Control Charts

The pioneering work of Hotelling (1947), who proposed the extension of the univariate Shewhart control chart to a chart that monitors the mean vector of multivariate processes, has inspired many scientists since then. Jackson (1985), Alt (1985) and Jarrett and Pan (2009) have, also, pointed out the need of simultaneous monitoring of several key quality characteristics of a process and proposed multivariate control charts. Mason and Young (2001) utilized the T^2 statistic for multivariate statistical process control. Chang and Bai (2004) and Yahaya *et al*. (2011) investigated alternative multivariate T^2 control charts. Tracy *et al*. (1992) proposed an exact method for the construction of control limits in multivariate control charts. Moreover, Nedumaran and Pignatiello (1999) studied the use of retrospective control limits on T^2 control charts. Bersimis *et al*. (2007) carried out a very detailed overview of multivariate control charts from the pioneering work of Hotelling (1947) until 2006.

2.6.1 Economic-Statistical Design of Multivariate Control Charts

Similarly to univariate control charts, after Duncan's (1956) innovative implementation of a cost-minimizing criterion for the design of a control chart, many researchers have investigated the economic and/or the economic-statistical design of a wide range of control charts for monitoring multivariate processes. Specifically, Montgomery and Klatt (1972) proposed a cost model for the design of a T^2 control chart. Molnau *et al*. (2001) presented the economic design of multivariate *EWMA* (*MEWMA*) control chart subject to specific statistical constraints. Moreover, Yang and Rahim (2005, 2006) proposed the economic-statistical design of a Hotelling's T^2 control chart when the occurrences of the assignable causes follow Weibull distributions with increasing failure rates. Yeong *et al*. (2014) studied the economic and economic-statistical design of a multivariate synthetic control chart.

2.6.2 Adaptive Multivariate Control Charts

The general conclusion that adaptive design parameters improve the performance of a control chart, has, also, been verified in multivariate T^2 control charts. Aparisi (1996) and Faraz and Moghadam (2009) presented Hotelling's T^2 control charts with adaptive sample sizes (*VSS*). Faraz *et al*. (2010) investigated the economic and economic-statistical design of *VSS* Hotelling's T^2 control charts. Furthermore, Hotelling's T^2 control charts that allow only the sampling interval to vary (*VSI*) have also been proposed over the years. Aparisi and Haro (2001), Chou *et al*. (2006) and Faraz *et al*. (2011) studied the design of multivariate *VSI* control charts through statistical and/or economic criteria. Chen (2007a) utilized the Markov chain theory to develop a cost model and genetic algorithms for the determination of the optimal values of the design parameters of a *VSI* T^2 control chart. Mahadik (2013) extended the *VSI* Hotelling's T^2 control chart to *VSIWL* chart, where, besides the sampling interval, an adaptive warning limit is utilized as well. Champ and Aparisi (2008) proposed two Double-Sampling (*DS*) Hotelling's T^2 control charts, whereas, He and Grigoryan (2005) proposed a Multiple-Sampling (*MS*) control chart for the simultaneous monitoring of multiple quality characteristics. In the work of Epprecht *et al*. (2013), a double-dimension T^2 (DDT^2) control chart is presented, where a number of p_1 out of p variables of the process are monitored and only in case the chart statistic falls in the warning area, the remaining p_2 variables $\left(p_2 = p - p_1\right)$ are combined with p_1 to redefine the T^2 statistic.

Chen and Hsieh (2007) investigated a Hotelling's T^2 control chart with adaptive sample sizes and adaptive control limits (*VSSC*), whereas, Aparisi and Haro (2003) and Chen (2009) with variable sampling sizes and sampling intervals (*VSSI*). Moreover, Faraz and Parsian (2006) extended the multivariate *VSSI* T^2 control chart to a *VSSI* T^2 chart that utilizes double warning lines (T^2 –*DWL*) and concluded that better shift detection ability is achieved. The same conclusion was derived by Faraz and Saniga (2011) who compared *FP*, *VSS*, *VSI*, *VSSI* and *MEWMA* control charts with the respective double warning line *VSSVSI* chart. Chen (2007b) proposed an economically optimized *VP* T^2 chart with a better shift detection ability compared to respective, less-adaptive, charts. Furthermore, Faraz *et al*. (2014) studied the double-objective economic-statistical design of a *VP* T^2 control chart.

2.6.3 Multivariate Control Charts for Monitoring Covariance Matrix

Another stream of research, focused on developing control charts that monitor shifts in the covariance matrix of a process. In the late 80s, Alt and Bedewi (1986), Healy (1987) and Alt and Smith (1988) studied shifts in the covariance matrix of processes, where multiple quality characteristics are monitored simultaneously. Guerrero-Cusumano (1995) proposed an interesting approach for testing the multivariate variability of a process through the measure of conditional entropy. Tang and Barnett (1996a, 1996b) presented some techniques for monitoring the dispersion of multivariate normal processes. Aparisi *et al*. (1999) investigated the statistical properties of a multivariate control chart that monitors the process dispersion through the generalized variance $|S|$. Moreover, Chan and Zhang (2001), Apley and Shi (2001), Levinson *et al*. (2002) and Khoo and Quah (2003) proposed control charts for monitoring the dispersion of multivariate processes. Aparisi *et al*. (2001) extended the design of the generalized variance chart to adaptive sampling sizes, whereas, Grigoryan and He (2005) to a multivariate Double-Sampling (*MDS*) control scheme. Yeh *et al*. (2003, 2004, 2005) investigated *MEWMA* control charts for monitoring process variability. A detailed review of the multivariate control charts that monitor the covariance matrix of a process, from 1990 to 2005, can be found in Yeh *et al*. (2006).

Huwang *et al*. (2007), Memar and Niaki (2009) and Yeh and Wang (2012) presented multivariate control charts for monitoring the process variability for individual observations. Furthermore, Hao *et al*. (2008) investigated a multivariate projection chart for monitoring process variability. Costa and Machado (2008) and Machado *et al*. (2009b) proposed the monitoring of the covariance matrix through maximum sample variance for bivariate processes while Costa and Machado (2009) extended their study to multivariate processes. Costa and Machado (2011a) concluded that the chart based on the maximum sample variance (*VMAX*) has a better performance compared to the chart based on maximum sample range (*RMAX*).

Recently, Liu *et al*. (2013) studied a multivariate synthetic control chart for monitoring the covariance matrix based on the combination of a conditional entropy chart and the conforming run length chart. Maboudou-Tchao and Agboto (2013) investigated the case where the number of observations is equal or even lower compared to the number of the monitored variables of the process, which normally leads the covariance matrix to singularity. Shen *et al*. (2014) proposed a new *MEWMA* control chart for monitoring covariance matrices, achieving enhanced statistical performance. Li *et al*. (2013) and Mitra and Clark (2014) also presented control charts for monitoring shifts in the covariance matrix of multivariate processes. Moreover, Lee and Khoo (2015) proposed a *VSI* multivariate synthetic control chart for shifts in the covariance matrix based on the generalized variance control chart.

2.6.4 Multivariate Control Charts for Joint Monitoring of Location and Scale

Few researchers have dealt with the problem of simultaneous monitoring of location and scale of processes where multiple quality characteristics should be monitored. Spiring and Cheng (1998) presented a single control chart for monitoring both location and scale of multivariate processes by plotting two variables in a single chart. Sullivan and Woodall (2000) studied a control chart for monitoring shifts in the mean vector, covariance matrix or both, when multivariate individual observations are available. Xie (1999) and Chen *et al*. (2005) proposed a single *MEWMA* control chart, whereas, Thaga (2004) a single multivariate *CUSUM* (*MCUSUM*) control chart. Chou *et al*. (2002, 2003) proposed the economic-statistical design of multivariate control charts for monitoring the mean vector and covariance matrix of processes, simultaneously. Yeh and Lin (2002) proposed a single two-dimensional control chart for the simultaneous monitoring of mean and variance of multivariate processes. Moreover, Khoo (2005) proposed a combination of the Hotelling's control chart and the generalized variance $|S|$ control chart for monitoring the mean vector and covariance matrix of processes where multiple correlated quality characteristics should be monitored.

Reynolds and Cho (2006) and Reynolds and Stoumbos (2008) proposed a variety of combinations of *MEWMA* and Shewhart control charts for the simultaneous monitoring of the mean vector and covariance matrix of processes and compared their performance. Reynolds and Cho (2011) extended this idea to adaptive sampling intervals in order to achieve better overall performance. Hawkins and Maboudou-Tchao (2008) combined two *EWMA* control charts for monitoring the location and the variability of a multivariate process and extended their research (Maboudou-Tchao and Hawkins, 2011) to monitoring processes from their start-up by re-calculation of the mean and standard deviation estimates, each time a sample is collected. Machado *et al*. (2009a) proposed a multivariate control chart based on the sample means and sample variances (*MVMAX*), whereas, Costa and Machado (2011b) proposed a control chart based on the standardized sample means and sample ranges (*MRMAX*) and implemented supplementary runs rules to achieve better performance (Costa and Machado, 2013). Furthermore, Zhang *et al*. (2010) investigated a single control chart for the simultaneous monitoring of the process mean and variability in multivariate cases, by combining the generalized likelihood ratio with the *EWMA* procedure. Recently, the simultaneous monitoring of the mean vector and covariance matrix was studied by Yang (2014) through a hybrid ensemble learning-based model, in bivariate processes, Wang *et al*. (2014) through the penalized likelihood estimation and Tasias and Nenes (2016c) through a conditional-entropy approach and for a multiple assignable cause mechanism affecting both the process location and scale.

3. PROBLEM SETTING AND ASSUMPTIONS

This chapter discusses the general assumptions that apply for every proposed control scheme in this dissertation. These model assumptions are divided into two categories depending on the type of the monitored process, i.e., univariate and multivariate. Additionally, a general framework of the formulation of the proposed schemes is provided. Finally, the method employed for the derivation of the optimum design parameters is presented.

3.1 Model Assumptions

3.1.1 Univariate Processes

A production process is assumed to operate for an infinite time horizon. The process is monitored through a critical quality characteristic *X*, which is normally distributed with target mean μ_T and target standard deviation σ_T. The initial process set-up is assumed to be perfect, that is, manufacturing operations always start with the population mean and standard deviation coinciding with the target $\left(\mu_0 = \mu_T, \sigma_0 = \sigma_T\right)$.

Multiple independent assignable causes may occur at random times affecting the mean (*m* assignable causes, $m > 0$) and/or the standard deviation (*r* assignable causes, $r > 0$) and shifting the process to out-of-control operating states. A process operation in any out-of-control state is undesirable, because it inevitably leads to poor quality output and/or higher operational cost. It is assumed that the larger the shift size, the poorer the quality output and the higher the operational costs. An assignable cause leading to a larger shift size can be identified more easily by the control scheme and eliminated.

Moreover, as the process is not self-correcting, only transitions to inferior states may occur. The occurrence of an assignable cause does not prevent the occurrence of another assignable cause, which can further deteriorate the process performance by shifting it to an inferior performance state, which is a state where the deviation from the target is larger, while the operational cost and the cost of removing the cause are also larger. Namely, *OOC* operation consists of different levels of "bad-quality" performance, depending on the assignable causes that affect the process mean and/or the standard deviation of the process.

Whenever an assignable cause occurs, the process remains under its effect until the occurrence is detected and its effect is removed or until a new deterioration occurs and the process is shifted to an inferior state. Furthermore, the assignable causes, which may occur independently, have a twofold negative effect on the process: not only do they shift either the process mean or the standard deviation or both away from target values, they also increase the occurrence rate of other assignable causes with a larger effect on the process, leading to progressive deterioration of the process performance.

A shift to a generic out-of-control mean μ_i is measured as a multiple δ_i of the standard deviation, that is $\mu_i = \mu_0 + \delta_i \cdot \sigma_0 \quad \left(i = 1, \ldots, m,\ \delta_i > \delta_{i-1} \geq 0\right)$. In a similar manner, the occurrence of an assignable cause *j* that affects the standard deviation of the process shifts the standard deviation from σ_0 to $\sigma_j = \gamma_j \cdot \sigma_0$ $\left(j = 1, \ldots, r,\ \gamma_j > \gamma_{j-1} \geq 1\right)$. The actual state of the process is defined by the assignable causes that may have occurred and is denoted by $\left(i, j\right)$ $\left(i \geq 0, j \geq 0\right)$.

The process is monitored by relatively easy to use, fully adaptive, Phase II, control schemes, proposed and studied in this dissertation for the first time. The proposed control schemes are determined by the values of the sample size *n*, the sampling interval *h* and the width of the warning limit coefficient *w* and control limit coefficient *k*, both for the chart that monitors the process mean $\left(w_x, k_x\right)$ and for the one that monitors the standard deviation of the process $\left(w_s, k_s\right)$. These design parameters $\left\{n_q, h_q, w_{x,q}, k_{x,q}, w_{s,q}, k_{s,q}\right\}$ are allowed to vary at two different levels, a relaxed $\left(q = 1\right)$ and a tightened one $\left(q = 2\right)$, where $n_2 \geq n_1$, $h_2 \leq h_1$, $w_{x,2} \leq w_{x,1} \leq k_{x,1}$, $w_{x,2} \leq k_{x,2} \leq k_{x,1}$, $w_{s,2} \leq w_{s,1} \leq k_{s,1}$ and $w_{s,2} \leq k_{s,2} \leq k_{s,1}$.

For ease of presentation, the process mean is monitored through the standardized sample mean, denoted by z_t, which is compared to the respective warning and control limit coefficients. On the other hand, each sample standard deviation, denoted by s_t, is compared to the upper warning and control limits of the control chart that monitors the standard deviation, denoted by $UWL_{s,q}$ and $UCL_{s,q}$, respectively. The upper warning and control limits are derived from $UWL_{s,q} = \left(c_{4,q} + w_s \cdot \sqrt{1 - c_{4,q}^2}\right) \cdot \sigma_0$ and $UCL_{s,q} = \left(c_{4,q} + k_{s,q} \cdot \sqrt{1 - c_{4,q}^2}\right) \cdot \sigma_0$, where $c_{4,q} = \left(\sqrt{2/(n_q - 1)}\right) \cdot \left(\Gamma\left(n_q / 2\right) / \Gamma\left((n_q - 1)/2\right)\right)$.

It should be noted that based on the assumption that *X* is a normally distributed random variable, the sample mean is also a normal variable $\overline{X} \sim N\left(\mu_i, \sigma_j^2 / n\right)$ with the value of its mean and its standard deviation depending on the actual state of the process. Obviously, the standardized sample mean follows a standard normal distribution with mean of zero and standard deviation of one $\left(z \sim N(0,1)\right)$ in the *IC* state. Moreover, in order to compute the probability of the sample standard deviation to be in one of the central, warning or action zones, a simple transformation to variable $X^2 = (n-1) \cdot s^2 / \sigma_j^2$, which is a continuous random variable following a chi-square distribution with $n-1$ degrees of freedom, is necessary.

The time to the occurrence of each quality shift follows a non-negative exponential distribution. The occurrence rates of the assignable causes are denoted by $\lambda_{x(i \to k)}$ $\left(\lambda_{s(j \to l)}\right)$ for a transition of the mean (standard deviation) from i to k, $k > i$ (*j* to *l*, $l > j$).

The transition rates to any inferior state, either for the mean or for the standard deviation of the process, when operating under quality shifts *i* and *j*, respectively, are denoted by $v_{x,i}$ and $v_{s,j}$ $\left(v_{x,i} = \sum_{k=i+1}^{m} \lambda_{x(i \to k)}, v_{s,j} = \sum_{l=j+1}^{r} \lambda_{s(j \to l)}\right)$.

3.1.2 Multivariate Processes

In multivariate processes, the quality of the process is fully characterized by *p* correlated variables. The process is assumed to operate indefinitely and to have a perfect initial set-up with $\mu_0' = \mu_T'$ and $\Sigma_0 = \Sigma_T$. Multiple independent assignable causes may occur according to a Poisson process, affecting either the mean vector and/or the covariance matrix.

A realistic assumption is that a transition of the process from one state to another is possible only if no quality characteristic is improved. In other words, no self-correction is allowed. Furthermore, similarly to the univariate case, when the process operates *OOC* due to the occurrence of an/some assignable cause/es, it remains under their effect until the occurrence is detected and appropriate corrective actions restore the process to the *IC* state, or until a new deterioration occurs and the process is shifted to an inferior state.

The quality characteristics are assumed to follow a *p*-variate normal distribution with known in-control mean vector $\mu_0' = \left(\mu_{0,1}, \mu_{0,2}, ..., \mu_{0,p}\right)$ and in-control covariance matrix $\Sigma_0 = \begin{bmatrix} \sigma_{0,1}^2 & \sigma_{0,1} \cdot \sigma_{0,2} & ... & \sigma_{0,1} \cdot \sigma_{0,p} \\ \sigma_{0,2} \cdot \sigma_{0,1} & \sigma_{0,2}^2 & ... & \sigma_{0,2} \cdot \sigma_{0,p} \\ ... & ... & ... & ... \\ \sigma_{0,p} \cdot \sigma_{0,1} & \sigma_{0,p} \cdot \sigma_{0,2} & ... & \sigma_{0,p}^2 \end{bmatrix}$.

The occurrence of an assignable cause *i* that affects the mean vector shifts the mean vector from μ_0' to $\mu_i' = \left(\mu_{i,1} + \delta_{i,1} \cdot \sigma_{0,1}, ..., \mu_{i,p} + \delta_{i,p} \cdot \sigma_{0,p}\right)$, without affecting the covariance matrix which is assumed to remain constant. On the other hand, the occurrence of an assignable cause *j* that affects the covariance matrix shifts only the covariance matrix away from its target value Σ_0 to

$$\Sigma_j = \begin{bmatrix} \gamma_{j,1}^2 \cdot \sigma_{j,1}^2 & \gamma_{j,1} \cdot \sigma_{0,1} \cdot \gamma_{j,2} \cdot \sigma_{0,2} & ... & \gamma_{j,1} \cdot \sigma_{0,1} \cdot \gamma_{j,p} \cdot \sigma_{0,p} \\ \gamma_{j,2} \cdot \sigma_{0,2} \cdot \gamma_{j,1} \cdot \sigma_{0,1} & \gamma_{j,2}^2 \cdot \sigma_{j,2}^2 & ... & \gamma_{j,2} \cdot \sigma_{0,2} \cdot \gamma_{j,p} \cdot \sigma_{0,p} \\ ... & ... & ... & ... \\ \gamma_{j,p} \cdot \sigma_{0,p} \cdot \gamma_{j,1} \cdot \sigma_{0,1} & \gamma_{j,p} \cdot \sigma_{0,p} \cdot \gamma_{j,2} \cdot \sigma_{0,2} & ... & \gamma_{j,p}^2 \cdot \sigma_{j,p}^2 \end{bmatrix}.$$

The correlations between the quality characteristics are assumed to remain constant and unaffected by the assignable causes. This is a common assumption in the literature made by many researchers (Costa and Machado 2011b; Celano *et al*. 2014; Chen *et al*. 2002).

Similarly to the univariate case, multivariate processes are monitored by a control scheme determined by the following design parameters $\{n_q, h_q, w_{mv,q}, k_{mv,q}, w_{cm,q}, , k_{cm,q}\}$, $q=1$ (2) for the relaxed (tightened) set, where $n_2 \geq n_1$, $h_2 \geq h_1$, $w_{mv,2} \leq \leq w_{mv,1} \leq k_{mv,1}$, $w_{mv,2} \leq k_{mv,2} \leq k_{mv,1}$, $w_{cm,2} \leq w_{cm,1} \leq k_{cm,1}$ and $w_{cm,2} \leq k_{cm,2} \leq k_{cm,1}$.

The sets of assignable causes that affect the mean vector and the covariance matrix are denoted by m_{mv} and r_{cm}, respectively. The occurrence rates of the assignable causes are denoted by $\lambda_{mv(i \to k)}$ $\left(\lambda_{cm(j \to l)}\right)$ for a transition of the mean vector (covariance matrix) from *i* to *k* (*j* to *l*). The transition rates to any inferior state either for the mean vector or for the covariance matrix of the process, when operating under assignable cause *i* and *j*, respectively, are denoted by $v_{mv,i}$ and $v_{cm,j}$ $\left(v_{mv,i} = \sum_{k \in \{1,2,\ldots,m_{mv}\} \setminus \{i\}} \lambda_{mv(i \to k)}, v_{cm,j} = \sum_{l \in \{1,2,\ldots,r_{cm}\} \setminus \{j\}} \lambda_{cm(j \to l)} \right)$.

It should be mentioned that in order for the covariance matrix to have full rank and be invertible, the sample size *n* should be greater than the number of the correlated quality characteristics p $(n > p)$.

3.2 Model Formulation

It is assumed that only by means of an on-line sampling procedure can the effect of an assignable cause be detected. Specifically, at each sampling instance a sample is collected from the process and, either the mean and standard deviation as regards univariate processes or the mean vector and covariance matrix as regards multivariate processes, are computed. Based on this information, decisions are made concerning not only the process, but also the control scheme itself. Specifically, it is decided whether the process must be stopped or not, what should the next sample size be, how long should the process operate until the next sampling instance and what should the values of warning and control limits be.

The answers to the above questions depend on the values of the statistics of the collected sample and the severity of the control procedure (relaxed or tightened). The values of the control limits and warning limit coefficients of the two control charts define three regions (zones) on the charts, the central, the warning and the action zone.

All the possible decisions at any sampling instance *t* are denoted by a_t and are defined as: (a) $a_t = 0$ if the process should not be interrupted; (b) $a_t = 1$ if there is a warning for the effect of some assignable cause and the control of the process has to be tightened; (c) $a_t = 2$ if an alarm is issued and the process must be stopped for investigation and possible restoration if an assignable cause has indeed occurred.

Whenever the control scheme indicates a possible out-of control operation of the process, an inspection takes place, in order to reveal any possible assignable cause's effect. Then, if an assignable cause has indeed occurred, the process is perfectly restored to the *IC* state ($\mu = \mu_0$ and $\sigma = \sigma_0$). It is assumed that the process does not continue its operation during search and repair. However, this assumption can be relaxed and the model can be easily modified to account for the case where the process continues its operation during search and repair.

Each assignable cause calls for a different corrective action, depending on the out-of-control state characterizing the process. If no assignable cause has occurred, the process, obviously, remains *IC*. Thus, after any signal of the scheme for an alarm, the process will resume its operation from the *IC* state.

As already mentioned, the actual state of the process at any sampling instance *t* is denoted by Y_t and is fully defined by the assignable causes that may have affected the process location and/or variability. Subsequently, $Y_t = (0,0)$ corresponds to the in-control condition, whereas $Y_t = \{\{(0,1),...,(0,r)\},\{(1,0),(1,1),...,(1,r)\},...,\{(m,0),(m,1),...,(m,r)\}\}$, refers to an out-of-control condition due to the occurrence of at least one assignable cause (for multivariate processes *m* refers to m_{mv} and *r* refers to r_{cm}).

A three-dimensional *DTMC* (Y_t, a_t) is utilized, exploiting the assumption that the time for the occurrence of an assignable cause is an exponentially distributed random variable, to describe both the actual state of the process and the decision made at each sampling instance. It should be mentioned that a special feature of this *DTMC* is that the duration of each transition step, i.e., the actual duration between the beginning of two successive sampling intervals, may be different in actual time units. This happens due to the variable sampling intervals and the time delays we have considered for searching and removing an assignable cause or revealing a false alarm, which will be hereafter discussed.

The general expression of the transition probabilities, which represent the probability for the process to operate in state $\left(Y_t, a_t\right)$ at the end of a transition step is based on Markov chain theory and is defined as follows:

$$P\left[a_t = v, Y_t = \left(k,l\right) \middle| a_{t-1} = u, Y_{t-1} = \left(i,j\right)\right] \text{ for } i,k \in \left[0,m\right], j,l \in \left[0,r\right], \text{ and } u, v = 0,1,2$$

The cost of lost production time for the process investigation and the cost of the time needed for the removal of an assignable cause are included in the costs of investigation and restoration, respectively. The time to search and restore the process from state $\left(i,j\right)$ is denoted by $T_{(i,j)}$. It should be noted that the expected time for the process restoration from an assignable cause is positively correlated with the effect of the assignable cause to the process ($T_{(k,l)} > T_{(i,j)} > 0$ for $k > i > 0$, $l > j > 0$). Finally, the time needed to reveal a false alarm is denoted by $T_{(0,0)}$. It is assumed that $T_{(0,0)} \le T_{(i,j)}$ $\left(i,j\right) \ne 0$.

The cost elements that have an economic impact on the process and arise from the process monitoring can be divided into four different categories: (a) the sampling costs, which can be divided into a fixed cost per sample *b* and a variable cost per sample unit *c*. For multivariate processes, the variable and fixed costs for sampling and testing are considered to be different for each quality characteristic and are denoted by c_ρ and b_ρ $\left(\rho = 1,2,...,p\right)$, respectively; (b) the cost per time unit of operation under the effect of assignable causes *i* and/or *j*, denoted by $M_{(i,j)}$; (c) the cost for the restoration of the process to the *IC* state by removing the effect of assignable causes *i* and *j* $\left(\left(i,j\right) \ne \left(0,0\right)\right)$, denoted by $L_{(i,j)}$; (d) the false alarm cost

$L_{(0,0)}$, when the production is erroneously stopped for investigation of the process for possible assignable causes (because no assignable cause has actually occurred). It is apparent that the lower the effect of an assignable cause to the process location and/or variability, the lower the $M_{(i,j)}$ and $L_{(i,j)}$ costs.

As already defined, the process starts *IC* and continues its operation until an *OOC* signal occurs and the process operation is halted for an inspection. This quality cycle, that consists of production, monitoring and inspection, is assumed to be a renewal-reward process till the actual process restoration.

The total quality-related cost per time unit, denoted by *ECT*, may be computed as the ratio of the expected cost of a transition step, denoted by *EC*, over its expected duration, denoted by ET $\left(ECT = EC / ET\right)$. The average cost (duration) of a transition step may be evaluated as the weighted average of the expected cost (duration) of each of the possible states of the process. The minimization of *ECT* defines the optimum design parameters and is also utilized as the basic measure of economic performance for the proposed models.

It is worth noting that the proposed models discussed in this thesis can be easily extended to any other partially adaptive respective control scheme.

3.3 Optimization Method

For the derivation of the optimum design parameters, it is assumed that the two warning limits of each control chart are the same. This assumption is made in order to simplify the proposed schemes and, is based on the research of Park and Reynolds (1999), who found the marginal cost reduction, i.e., the reduction in the total cost when the number of warning limits is incremented by one, too small to justify the extra complexity. It should be noted that this does not affect in any way neither the description of the proposed control schemes, nor the preceded model development.

In order to define the optimum set of design parameters, denoted by DP_q, for each of the proposed control schemes, an exhaustive, two-step algorithm has been developed. First, a coarse discretization is used to define the allowable values of each parameter. In particular, each design parameter is allowed to vary in 0.5 increments within a pre-specified range, with the exception of the sample sizes $(n_1 \text{ and } n_2)$, for which we use an increment of 5. The minimum allowable values of the design parameters are: $\left[h_1, h_2, n_1, n_2, w_x, k_{x,1}, k_{x,2}, w_s, k_{s,1}, k_{s,2}\right]_{\min} = [0.1, 0.1, 2, 2, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1]$, whereas, the set of the respective maximum allowable values is the following: $\left[h_1, h_2, n_1, n_2, w_x, k_{x,1}, k_{x,2}, w_s, k_{s,1}, k_{s,2}\right]_{\max} = [7.1, h_1, n_2, 32, k_{x,2}, 5.1, k_{x,1}, k_{s,2}, 5.1, k_{s,1}]$. After the derivation of the near-optimum parameters of the first step (with the coarse discretization), a more fine discretization is used. In particular, we examine all values of $h_1, h_2, w_x, k_{x,1}, k_{x,2}, w_s, k_{s,1}, k_{s,2}$ in the range of ±0.5 around their respective near-optimum values of the first step of the algorithm, with an increment of 0.1, and all values of n_1, n_2 in the range of ±5 with an increment of 1. Of course, these ranges are further limited by the conditions $0.1 \le h_2 \le h_1$, $2 \le n_1 \le n_2$, $0.1 \le w_x \le k_{x,2} \le k_{x,1}$ and $0.1 \le w_s \le k_{s,2} \le k_{s,1}$. Obviously, regarding multivariate processes $w_x, k_{x,1}, k_{x,2}, w_s, k_{s,1}, k_{s,2}$ should be substituted by $w_{mv}, k_{mv,1}, k_{mv,2}$, $w_{cm}, k_{cm,1}, k_{cm,2}$, respectively.

3.4 Statistical Measures

The model described above implements just one transition probability matrix to fully represent the process operation. Thus, it can be easily used not only to get the economic design of any adaptive control scheme but also to estimate several important measures of statistical performance. The values of these statistical measures can be evaluated, given the proposed Markov chain formulation, through the steady-state probabilities of the process.

- <u>*Type I* error</u> α

In the conventional Shewhart chart, the *Type I* error probability (α) is defined as the probability of issuing an alarm, given that the process operates in statistical control. Given the proposed Markov chain model formulation, the *Type I* error probability (α) is equal to the ratio of the probability of issuing a false alarm $\left(\pi_{(0,0)2}\right)$ over the probability of operating in-control in the long run $\left(\pi_{(0,0)0} + \pi_{(0,0)1} + \pi_{(0,0)2}\right)$:

$$\alpha = \frac{\pi_{(0,0)2}}{\pi_{(0,0)0} + \pi_{(0,0)1} + \pi_{(0,0)2}} \tag{3.1}$$

- Average number of false alarms per time unit *ANOF*

The average number of false alarms per time unit (*ANOF*), can be computed as the ratio of the probability of issuing a false alarm over the average length of a transition step:

$$ANOF = \frac{\pi_{(0,0)2}}{ET} \tag{3.2}$$

- Average power of the chart $1-\beta$

The power of a conventional Shewhart chart $(1-\beta)$ is defined as the probability of issuing an alarm, given that the process actually operates under the effect of an assignable cause. With reference to the multiple assignable cause scenario, the power of the chart $\left(1-\beta_{(i,j)}\right)$ vs. the assignable causes leading to state (i, j) is given by the ratio of the probability of issuing an alarm $\pi_{(i,j)2}$ over the probability of operating out-of-control $\left(1-\left(\pi_{(0,0)0}+\pi_{(0,0)1}+\pi_{(0,0)2}\right)\right)$.

$$1-\beta_{(i,j)} = \frac{\pi_{(i,j)2}}{1-\left(\pi_{(0,0)0}+\pi_{(0,0)1}+\pi_{(0,0)2}\right)} \tag{3.3}$$

It should be noted, however, that the above probability varies with (i, j) and is expected to increase as the magnitude of shifts, i.e., δ_i and γ_j, get larger values. Thus, a more appropriate measure to estimate the overall power of the chart is the weighted average power, denoted by $1-\beta$, which can be computed as:

$$1-\beta=\sum_{(i,j)\neq 0}\left(1-\beta_{(i,j)}\right)\cdot$$

$$\cdot \mathrm{P}\left(\text{out of control operation with } \delta=\delta_i \text{ and } \gamma=\gamma_j \middle| \text{out of control operation}\right) \quad (3.4)$$

$$=\sum_{(i,j)\neq 0}\left(\frac{\pi_{(i,j)2}}{\pi_{(i,j)0}+\pi_{(i,j)1}+\pi_{(i,j)2}}\cdot\frac{\pi_{(i,j)0}+\pi_{(i,j)1}+\pi_{(i,j)2}}{1-\left(\pi_{(0,0)0}+\pi_{(0,0)1}+\pi_{(0,0)2}\right)}\right)$$

and thus,

$$1-\beta=\sum_{(i,j)\neq(0,0)}\left(\frac{\pi_{(i,j)2}}{1-\left(\pi_{(0,0)0}+\pi_{(0,0)1}+\pi_{(0,0)2}\right)}\right) \tag{3.5}$$

➢ <u>In-control average run length</u> ARL_0

The in-control average run length (ARL_0) can be computed by the well-known following function of α:

$$ARL_0=\frac{1}{a}=\frac{\pi_{(0,0)0}+\pi_{(0,0)1}+\pi_{(0,0)2}}{\pi_{(0,0)2}} \tag{3.6}$$

➢ <u>Weighted out-of-control average run length</u> $WARL$

Similarly, it is possible to estimate an expected out-of-control average run length ($WARL$), which is defined here as the weighted average number of samples that are needed to have a chart signal revealing the occurrence of any assignable cause:

$$WARL=\frac{1}{1-\beta}=\left(\frac{1-\left(\pi_{(0,0)0}+\pi_{(0,0)1}+\pi_{(0,0)2}\right)}{\sum_{(i,j)\neq(0,0)}\pi_{(i,j)2}}\right) \tag{3.7}$$

➢ <u>Expected time between two successive detections of the assignable cause</u> ATC

Another statistical measure of performance that might be of interest to the practitioners is the expected time between two successive detections of assignable causes (ATC), which can be viewed as the analogue to a cycle in the traditional case of renewal reward processes. It can be computed as the ratio of the average length of a transition step, *ET*, over the sum of the probabilities of detecting the actual occurrence of any assignable cause:

$$ATC = \frac{ET}{\sum_{(i,j)\neq(0,0)} \pi_{(i,j)2}} \tag{3.8}$$

- <u>Expected time between the occurrence and the removal of an assignable cause</u> *EATR*

If the expected time of in-control operation $\left(1/\left(v_{x,0}+v_{s,0}\right)\right)$ is subtracted from *ATC*, then the expected time from the occurrence until the detection of an assignable cause *EATR* is estimated as:

$$EATR = ATC - \frac{1}{\left(v_{x,0}+v_{s,0}\right)} = \frac{ET}{\sum_{(i,j)\neq(0,0)} \pi_{(i,j)2}} - \frac{1}{\left(v_{x,0}+v_{s,0}\right)} \tag{3.9}$$

As stated above, *EATR* includes the time to remove the assignable cause, i.e., it is the average time from the occurrence of some assignable cause until the time that the process resumes its operation from the in-control state. In case of negligible times to search and remove the assignable causes, *EATR* essentially reduces to the Expected Average Time to Signal (*EATS*).

The statistical measures play a key twofold role in the proposed control schemes. Firstly, acceptable statistical performance of each and every one of the proposed control schemes is assured through specific statistical constraints. Moreover, they may be utilized to monitor the statistical performance and to statistically compare different schemes.

The choice of the exact value of an upper or lower bound set to a statistical measure should be made on a case by case basis and after careful scrutinization according to the monitored process in order to assure, apart from acceptable statistical performance, feasibility in real production applications, as well. In the optimization problem section of every chapter, the reader can find the non-binding choice of statistical constraints utilized for each control scheme.

3.5 Possible Extensions

3.5.1 General Features

It is worth noting that the general proposed problem setting, which accounts for multiple assignable causes affecting the mean and the standard deviation, may be utilized to monitor a process in various ways, after the modification of the aforementioned assumptions.

Specifically, through some simple modifications the applicability of the proposed model can be enhanced to monitor processes where: (a) a continuous deterioration mechanism is present; (b) in case of a single quality shift but multiple assignable causes that scale-up the occurrence rate of this shift; (c) maintenance actions are, also, required:

(a) Our problem setting can be used as a way to model the ageing process of the equipment in various ways. It could be assumed that the transition of the process mean and standard deviation from *IC* state to the assignable cause with the larger effect, let's say *m* and *r*, is only possible through the operation under every possible assignable cause, i.e., $\lambda_{x(i\to k)} = 0$ and $\lambda_{s(j\to l)} = 0$ for every $k > i+1$ and $l > j+1$ and only $\lambda_{x(i\to i+1)} > 0$, $\lambda_{s(j\to j+1)} > 0$. In such case, every possible state from $(0,0)$ to $(m-1, r-1)$ are transient states and (m,r) is the absorbing state. Under this assumption, the proposed models are simplified to monitor a process with a continuous deterioration mechanism, for example the ageing process of a working tool, without allowing any "jumps" of the mean or the standard deviation.

(b) The proposed models can be modified to the single assignable cause scenario, where only one assignable cause may affect the mean *m* ($\delta_i = 0$, $i = 1,...,m-1$, $\delta_m > 0$) and one the standard deviation *r* ($\gamma_j = 1$, $j = 1,...,r-1$, $\gamma_r > 1$), but every intermediate state between $(0,0)$ and (m,r) is assumed to increase the probability of the shift. This way, the shift is assumed to follow a generalized Erlang distribution.

(c) The proposed model, in the single assignable cause scenario, can be also utilized (with some modifications) as a maintenance model, by considering the *IC* state as the only healthy state, the (m,r) state as the only failure state and every intermediate state as deteriorated states. In such case, if the model issues an alarm and the process is in a deteriorated state, preventive maintenance actions are carried out to restore the process to the healthy state and, therefore, reduce the probability of the process transition to the failure state. Moreover, it is also interesting to note that all intermediate states are allowed to have different operating costs. Even in the simpler case, where the intermediate states are not associated with increased operational costs ($M_{(i,j)} = 0$ for all $i < m$ and $j < r$), signals issued by the chart while operating in those states, are not considered as false alarms. On the contrary, these alarms signify interventions which have actually a preventive maintenance character since they restore the process to state zero and reduce the probability of (or equivalently increase the expected time to) the occurrence of the final assignable cause. This possible extension of the proposed problem setting to an integrated maintenance and quality control scheme is comprehensively presented in Chapter 8.

3.5.2 Imperfect Process Restoration

As mentioned earlier, if an alarm is issued by the control scheme $(a_t = 2)$, then, the process is stopped and, if an assignable cause has indeed occurred, perfectly restored to the *IC* state. However, the restoration to the *IC* state may not be perfect. Specifically, it may be possible for the process to continue its operation under the effect of an assignable cause, even after the process has been stopped, the specific assignable cause has been detected and the restoration of the process has been attempted. The extension of the proposed model to cases where restoration to the *IC* state may not be perfect becomes even more important, because of the indiscernible restoration of assignable causes that affect the standard deviation, whenever it is needed.

Under the assumption of imperfect process restoration, the process may be restored from any state $(i,j) \neq (0,0)$ to any other superior state (u,v) according to given probabilities, which are denoted by $q_{\substack{(i,j)\\(u,v)}}$:

$$q_{\substack{(i,j)\\(u,v)}} = \begin{cases} q_{\substack{(i,j)\\(u,v)}} & (u \leq i) \cap (v \leq j) \\ 0 & (u > i) \cup (v > j) \end{cases} \tag{3.10}$$

Apparently, $\sum_{u=0}^{i} \sum_{v=0}^{j} q_{\substack{(i,j)\\(u,v)}} = 1$ and $q_{\substack{(0,0)\\(0,0)}} = 1$.

This general assumption for imperfect process restoration is realistic in many production processes and the restoration probabilities may be estimated from statistical data upon a Phase I analysis of the monitored process. A perfect process restoration policy can be easily accounted by setting $q_{\substack{(i,j)\\(0,0)}} = 1$ and $q_{\substack{(i,j)\\(u,v)}} = 0$ for any $(u,v) \neq (0,0)$.

This extension of the model modifies neither the operation of the control schemes nor the Markov chain that models each control scheme's operation. Nevertheless, it affects the values of some of the transition probabilities, so as to account for the probability of imperfect restoration. The affected transition probabilities are those corresponding to a true alarm, namely the ones where $a_t = 2$ and $Y_t \neq (0,0)$: only when the process is operating within one of these transition states, may an imperfect restoration take place.

A detailed implementation of imperfect process restoration in case of an alarm has been developed for the most sophisticated proposed model where multiple assignable causes and failures may occur and is presented in Chapter 8, Section 8.5. It is obvious that, in a similar and even simpler manner, each of the proposed control schemes could be modified to account for the realistic assumption of imperfect process restoration.

3.5.3 Two-Sided Control Charts

The proposed control schemes for univariate processes can be easily extended to cases where the assignable causes reduce the process mean $\left(\mu_i = \mu_0 + \delta_i \cdot \sigma_0, \delta_i < 0\right)$. On the other hand, for multivariate processes, a one-sided chart for the process location is considered, because the mean vector is monitored through the distance of the sample mean vector $\bar{x}'$ from the target mean vector $\mu_0{}'$ and only upward shifts constitute deterioration of the process performance, as it will be explained in detail in Chapter 9.

It should be noted here that, regarding the assignable causes that may affect the standard deviation, we deliberately leave them to cause only upward shifts of the process standard deviation $\left(\gamma > 1\right)$ since any downward shifts $\left(\gamma < 1\right)$ would essentially mean a process improvement. It is thus reasonable to consider only one-sided *s* charts since in most real cases it would be unreasonable to consider that the process may improve by itself and try to "capture" this by adding lower control (and warning) limits in the *s* chart. In a similar manner, a downward shift of the covariance matrix would, unreasonably for our model, imply a self-correction of the process.

The detection of an assignable cause that may either increase or decrease the process mean, necessitates the use of a two-sided control chart. The extension of the model to its two-sided version does not affect the operation of the control schemes, as it has been described previously. However, the transition probabilities of the respective Markov chain need to be slightly altered to incorporate the two-sided effect of the assignable causes that affect the process mean. It is apparent that the value of each transition probability would essentially remain the same for both positive $\left(\delta_i > 0\right)$ and negative shifts $\left(\delta_i < 0\right)$ due to the symmetry of the normal distribution around μ_i.

4. $VP\ \overline{X}-s$ CONTROL SCHEME (VP_1)

4.1 Introduction

In this chapter, a new economic-statistically designed, fully adaptive control scheme is developed for processes where two independent assignable causes may occur, which affect the mean $(m=1)$ and/or the standard deviation $(r=1)$ of the distribution of the quality characteristic to be monitored.

This chapter is organized as follows. In Section 4.2 the stochastic model that describes the operation of the control scheme is presented. In Section 4.3 the cost model is constructed. Section 4.4 describes the optimization problem and Section 4.5 provides an illustrative example. Finally, in Section 4.6 a numerical investigation is carried out and performance comparisons against less sophisticated control schemes are performed.

It should be noted that this chapter is based on the paper of Tasias and Nenes (2012).

4.2 Mathematical Model

A fully adaptive Shewhart-type $VP\ \overline{X}-s$ control scheme, denoted by VP_1, is utilized to monitor the process. The control scheme's operation can be summarized as follows:

- If both the standardized mean of the collected sample and its standard deviation are plotted within the "so-called" central regions of the control charts, which essentially means that both statistics are below the warning limits (and of course could even be below the central lines of the charts), then a decision is taken to let the process continue its operation $(a_t=0)$ and a relaxed sampling is scheduled at the next inspection. Thus, the next sample size will be n_1 and the next sampling interval should be h_1. Finally, the warning and action control limit coefficients of the two control charts should be $w_{x,1}, w_{s,1}, k_{x,1}$ and $k_{s,1}$, because the process seems to operate in statistical control.

- If the standardized sample mean, or the sample standard deviation, or both of them fall within the warning regions of the control charts, but none of them is in the action region, then there is a warning that an assignable cause has occurred and thus, $a_t = 1$. Therefore, the control scheme switches to a different state, named warning state, and the control gets more tightened by waiting less time before taking the next sample: in fact the next inspection is scheduled after h_2 time units $(h_2 \leq h_1)$, the size of the next sample is increased to n_2 $(n_2 \geq n_1)$ and the action limit coefficients for the next sampling get narrower $(k_{x,2} \leq k_{x,1}, k_{s,2} \leq k_{s,1})$.

- Finally, when a standardized sample mean, or sample standard deviation, or both of them fall in the action region of the two control charts, an alarm is issued $(a_t = 2)$ and the production run is stopped. An investigation reveals with certainty the process status and if an assignable cause has actually occurred, a restoration to *IC* state takes place. Assuming that the repair procedure is perfect, the process is always restored in the *IC* state and the next sample's parameters will be $n_1, h_1, w_x, k_{x,1}, w_s, k_{s,1}$.

The regions described above in the two one-sided control charts are illustrated in Figure 4-1. The dotted lines depict the tightened warning and control limits and the arrows (dotted or not) the regions.

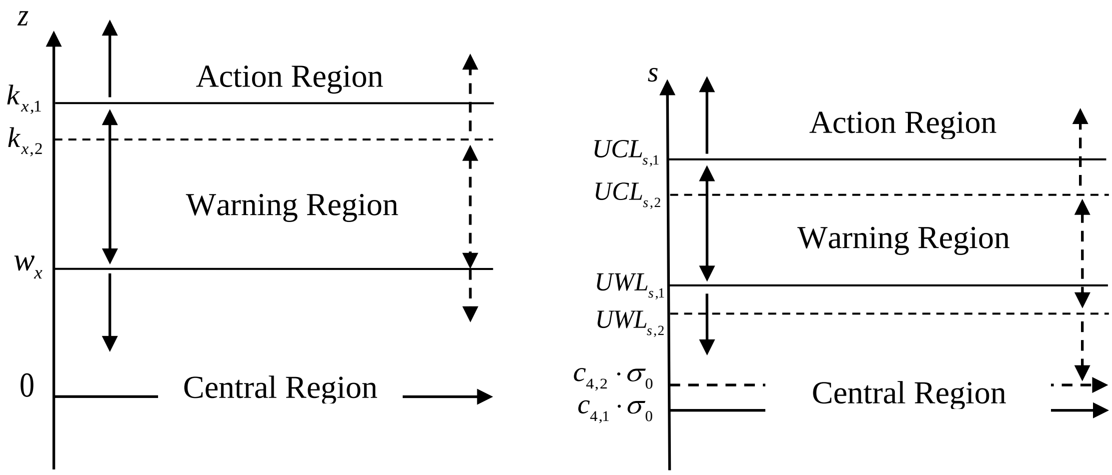


Figure 4-1: Regions of the VP_1 control scheme

The two assignable causes are not correlated and affect the process independently: thus, the process may operate in any of the four following states, at any sampling instance *t*. Specifically, if the process operates in statistical control, i.e., when $\mu = \mu_0$ and $\sigma = \sigma_0$, then $Y_t = (0,0)$. In the same way, $Y_t = (0,1)$ when $\mu = \mu_0$ and $\sigma = \sigma_1$, $Y_t = (1,0)$ when $\mu = \mu_1$ and $\sigma = \sigma_0$ and $Y_t = (1,1)$ when $\mu = \mu_1$ and $\sigma = \sigma_1$.

Given the possible values of each of the two components that constitute the three-dimensional state, $Y_t = \{(0,0),(0,1),(1,0),(1,1)\}$ and $a_t = 0,1,2$, a 12×12 transition probability matrix P can be formed as follows.

$$
\begin{array}{ccc}
 & (k,l) & \begin{array}{cccccccccccc} \multicolumn{3}{c}{(0,0)} & \multicolumn{3}{c}{(0,1)} & \multicolumn{3}{c}{(1,0)} & \multicolumn{3}{c}{(1,1)} \end{array} \\
 & a_t & \begin{array}{cccccccccccc} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \end{array} \\
(i,j) & a_{t-1} & \\
 & \begin{array}{c} 0 \\ 1 \\ 2 \\ 0 \\ 1 \\ 2 \\ 0 \\ 1 \\ 2 \\ 0 \\ 1 \\ 2 \end{array} & \\
\end{array}
$$

$$
P = \begin{array}{cc}
\begin{array}{c} \\ (0,0) \\ \\ \\ (0,1) \\ \\ \\ (1,0) \\ \\ \\ (1,1) \\ \\ \end{array}
&
\begin{array}{c} 0 \\ 1 \\ 2 \\ 0 \\ 1 \\ 2 \\ 0 \\ 1 \\ 2 \\ 0 \\ 1 \\ 2 \end{array}
\left[\begin{array}{c|c|c|c}
 & & & \\ \hline
 & & & \\ \hline
 & & & \\ \hline
 & & &
\end{array}\right]
\end{array}
$$

Figure 4-2: Transition Probability Matrix of the VP_1 control scheme

In order to compute the transition probabilities of the transition probability matrix P, every probability for the process moving from any state to any other state, must be first computed. Based on the fact that the two assignable causes affect independently the mean and the standard deviation of the process, the probability of a transition from any state (i,j) to any other possible state (k,l), can be computed from the following expressions:

$$p_{\substack{(0,0)\\(0,0)}}(h_q) = \exp\left(-\lambda_{x(0\to1)} \cdot h_q\right) \cdot \exp\left(-\lambda_{s(0\to1)} \cdot h_q\right) \tag{4.1}$$

$$p_{\substack{(0,0)\\(0,1)}}(h_q) = \exp\left(-\lambda_{x(0\to1)} \cdot h_q\right) \cdot \left(1 - \exp\left(-\lambda_{s(0\to1)} \cdot h_q\right)\right) \tag{4.2}$$

$$p_{\substack{(0,0)\\(1,0)}}(h_q) = \left(1 - \exp\left(-\lambda_{x(0\to1)} \cdot h_q\right)\right) \cdot \exp\left(-\lambda_{s(0\to1)} \cdot h_q\right) \tag{4.3}$$

$$p_{\substack{(0,0)\\(1,1)}}(h_q) = \left(1 - \exp\left(-\lambda_{x(0\to1)} \cdot h_q\right)\right) \cdot \left(1 - \exp\left(-\lambda_{s(0\to1)} \cdot h_q\right)\right) \tag{4.4}$$

$$p_{\substack{(0,1)\\(0,1)}}(h_q) = \exp\left(-\lambda_{x(0\to1)} \cdot h_q\right) \tag{4.5}$$

$$p_{\substack{(0,1)\\(1,1)}}(h_q) = 1 - \exp\left(-\lambda_{x(0\to1)} \cdot h_q\right) \tag{4.6}$$

$$p_{\substack{(1,0)\\(1,0)}}(h_q) = \exp\left(-\lambda_{s(0\to1)} \cdot h_q\right) \tag{4.7}$$

$$p_{\substack{(1,0)\\(1,1)}}(h_q) = 1 - \exp\left(-\lambda_{s(0\to1)} \cdot h_q\right) \tag{4.8}$$

The exact expressions for all the transition probabilities are equal to the product of the probability of the transition of the actual state of the process (equation (4.1) to (4.8)) times the probability to make the decision a_t $(a_t = 0,1,2)$ at sampling instance t and can be computed from the following expressions (a_{t-1} defines whether relaxed or tightened parameters have been utilized (for $a_{t-1} = 0(1)$ $q = 1(2)$). It should be mentioned that the probability of any transition, if no investigation takes place, is computed only for cases where $(i \le k) \cap (j \le l)$ and equals zero otherwise $((i > k) \cup (j > l))$:

$$Prob_{\substack{(i,j)a_{t-1}\\(k,l)a_t}}\left(h_q\right)=\begin{cases} p_{\substack{(i,j)\\(k,l)}}\left(h_q\right)\cdot\Phi\left(\dfrac{w_x-\delta_k\sqrt{n_q}}{\gamma_l}\right)\cdot P\left(\chi^2_{n_q-1}<\left(\dfrac{UWL_{s,q}}{\gamma_l\cdot\sigma_0}\right)^2\cdot\left(n_q-1\right)\right) & a_t=0\\ p_{\substack{(i,j)\\(k,l)}}\left(h_q\right)\cdot\left[\begin{array}{l}\Phi\left(\dfrac{k_{x,q}-\delta_k\sqrt{n_q}}{\gamma_l}\right)\cdot P\left(\chi^2_{n_q-1}<\left(\dfrac{UCL_{s,q}}{\gamma_l\cdot\sigma_0}\right)^2\cdot\left(n_q-1\right)\right)-\\ -\Phi\left(\dfrac{w_x-\delta_k\sqrt{n_q}}{\gamma_l}\right)\cdot P\left(\chi^2_{n_q-1}<\left(\dfrac{UWL_{s,q}}{\gamma_l\cdot\sigma_0}\right)^2\cdot\left(n_q-1\right)\right)\end{array}\right] & a_t=1\\ p_{\substack{(i,j)\\(k,l)}}\left(h_q\right)\cdot\left[1-\Phi\left(\dfrac{k_{x,q}-\delta_k\sqrt{n_q}}{\gamma_l}\right)\cdot P\left(\chi^2_{n_q-1}<\left(\dfrac{UCL_{s,q}}{\gamma_l\cdot\sigma_0}\right)^2\cdot\left(n_q-1\right)\right)\right] & a_t=2\end{cases}\tag{4.9}$$

It should be noted that the transition probabilities of the 3rd, 6th, 9th and 12th line of the transition probability matrix P (see Figure 4-2), are equal to the respective transition probabilities of the first line of P. The reason is that each time the production run is stopped for investigation and possible restoration, namely every time there is a standardized sample mean, sample standard deviation, or both, falling within the action region, the process is interrupted and restored (if needed) certainly, to the *IC* state. As a result, the probability thereafter of a transition to every possible state from each of the above states equals the respective probability of transition from the *IC* state.

$$Prob_{\substack{(i,j)2\\(k,l)a_t}}\left(h_q\right)=Prob_{\substack{(0,0)0\\(k,l)a_t}}\left(h_1\right)\tag{4.10}$$

The steady-state probabilities, which represent the long-term probability for the process being in state $\left(Y_t,a_t\right)$ are denoted by $\pi_{Y_t a_t}$ and are computed by solving the following linear system:

$$\pi_{Y_t a_t}=\sum_{Y_{t-1}=(0,0)}^{(1,1)}\sum_{a_{t-1}=0}^{2}Prob_{\substack{Y_{t-1}a_{t-1}\\Y_t\ a_t}}\left(h_q\right)\cdot\pi_{Y_{t-1}a_{t-1}}\ \text{ and }\ \sum_{Y_t=(0,0)}^{(1,1)}\sum_{a_t=0}^{2}\pi_{Y_t a_t}=1\tag{4.11}$$

Note that the idea of using the steady-state probabilities to describe both the condition of the process and the decision indicated by the control scheme has also been used for *VP* $\overline{X}$ charts in Nenes (2011) while it was first introduced in Nenes and Tagaras (2008) for fixed-parameter charts.

4.3 The Economic-Statistical Design

In order to compute the average cost of a transition step *EC*, we first need to compute the expected *OOC* operation cost of the process, starting from state (i, j) at the beginning of a sampling interval of duration h_q, denoted by $K_{(i,j)}(h_q)$.

For the computation of $K_{(i,j)}(h_q)$, the following parameters should be computed:

(i) The cost per time unit if no assignable cause occurs and the process remains under the effect of state (i, j) $\left(M_{(i,j)}\right)$ for the whole interval, multiplied by the probability that no assignable cause occurs within the interval $\left(\exp\left(-\left(v_{x,i} + v_{s,j}\right) \cdot h_q\right)\right)$, multiplied by the duration h_q.

(ii) The cost per time unit if only the process mean (standard deviation) is shifted, multiplied by the probability that such a scenario occurs, times the expected duration the process remains under the effect of state $(0, j)$ and $(1, j)$ $((i,0)$ and $(i,1))$.

(iii) The cost per time unit if both the mean and the standard deviation are shifted within a transition step. In this case, two scenarios are possible to occur: the assignable cause that affects the process mean to occur first and then the assignable cause that affects the standard deviation and vice versa.

The expected time of the occurrence of each assignable cause when both of them occur within a transition step, denoted by $\tau^{(1)}$ for the precedent assignable cause and by $\tau^{(2)}$ for the second one, have been discussed and analyzed in Nenes and Panagiotidou (2013) and are computed from the following expressions:

$$\tau^{(1)}\left(h_q\right) = \frac{\int_0^{h_q} t_1 \cdot \lambda_1 \cdot \exp\left(-\left(\lambda_{x(0\to1)} + \lambda_{s(0\to1)}\right) \cdot t_1\right) \int_{t_1}^{h_q} \lambda_2 \cdot \exp\left(-\left(v_{x,(1)} + v_{s,(1)}\right) \cdot \left(t_2 - t_1\right)\right) dt_2 dt_1}{\int_0^{h_q} \lambda_1 \cdot \exp\left(-\left(\lambda_{x(0\to1)} + \lambda_{s(0\to1)}\right) \cdot t_1\right) \int_{t_1}^{h_q} \lambda_2 \cdot \exp\left(-\left(v_{x,(1)} + v_{s,(1)}\right) \cdot \left(t_2 - t_1\right)\right) dt_2 dt_1} \tag{4.12}$$

$$\tau^{(2)}\left(h_q\right)=\frac{\int_0^{h_q}\lambda_1\cdot\exp\left(-\left(\lambda_{x(0\to1)}+\lambda_{s(0\to1)}\right)\cdot t_1\right)\int_{t_1}^{h_q}t_2\cdot\lambda_2\cdot\exp\left(-\left(v_{x,(1)}+v_{s,(1)}\right)\cdot\left(t_2-t_1\right)\right)dt_2dt_1}{\int_0^{h_q}\lambda_1\cdot\exp\left(-\left(\lambda_{x(0\to1)}+\lambda_{s(0\to1)}\right)\cdot t_1\right)\int_{t_1}^{h_q}\lambda_2\cdot\exp\left(-\left(v_{x,(1)}+v_{s,(1)}\right)\cdot\left(t_2-t_1\right)\right)dt_2dt_1}$$

(4.13)

It should be noted that in case the first assignable cause affects the process mean, then λ_1 should be substituted by $\lambda_{x(0\to1)}$ and λ_2 by $\lambda_{s(0\to1)}$. The exact opposite, i.e., the substitution of λ_1 by $\lambda_{s(0\to1)}$ and λ_2 by $\lambda_{x(0\to1)}$, should be made in case the first assignable cause affects the standard deviation. Obviously, the intermediate state would be $v_{x,(1)}=v_{x,1}$, $v_{s,(1)}=v_{s,0}$ in the first case, i.e., the first assignable cause shifts the process mean from its target value, and $v_{x,(1)}=v_{x,0}$, $v_{s,(1)}=v_{s,1}$ in the second case.

In a similar manner, the probability of each of the two scenarios to occur within a sampling interval of duration h_q, given that both assignable causes occur within the interval, is derived from the following equation:

$$\Pr\left(h_q,2\right)_{\substack{(0,0)\\(1,1)}}=\int_0^{h_q}\lambda_1\cdot\exp\left(-\left(\lambda_{x(0\to1)}+\lambda_{s(0\to1)}\right)\cdot t_1\right)\int_{t_1}^{h_q}\lambda_2\cdot\exp\left(-\left(v_{x,(1)}+v_{s,(1)}\right)\cdot\left(t_2-t_1\right)\right)dt_2dt_1$$

(4.14)

Subsequently, by denoting $M^{(1)}$ the cost per time unit when the process operates under the effect of the intermediate state, i.e., $(0,1)$ or $(1,0)$, $K_{(0,0)}\left(h_q\right)$ can be, now, computed as:

$$\begin{aligned}K_{(0,0)}\left(h_q\right)=&\exp\left(-\lambda_{s(0\to1)}\cdot h_q\right)\cdot\int_0^{h_q}\lambda_{x(0\to1)}\cdot\exp\left(-\lambda_{x(0\to1)}\cdot t\right)\cdot\left(t\cdot M_{(0,0)}+M_{(1,0)}\cdot\left(h_q-t\right)\right)dt+\\&+\exp\left(-\lambda_{x(0\to1)}\cdot h_q\right)\cdot\int_0^{h_q}\lambda_{s(0\to1)}\cdot\exp\left(-\lambda_{s(0\to1)}\cdot t\right)\cdot\left(t\cdot M_{(0,0)}+M_{(0,1)}\cdot\left(h_q-t\right)\right)dt+\\&+\Pr\left(h_q,2\right)_{\substack{(0,0)\\(1,1)}}\cdot\left[M_{(0,0)}\cdot\tau^{(1)}+M^{(1)}\cdot\left(\tau^{(2)}-\tau^{(1)}\right)+M_{(1,1)}\cdot\left(h_q-\tau^{(2)}\right)\right]\end{aligned}$$

(4.15)

In a similar manner, $K_{(1,0)}\left(h_q\right)$ and $K_{(0,1)}\left(h_q\right)$ are derived from the following equations:

$$\begin{aligned} K_{(1,0)}\left(h_q\right) &= M_{(1,0)} \cdot h_q \cdot \exp\left(-\lambda_{s(0\to1)} \cdot h_q\right) + \\ &+ \int_0^{h_q} \lambda_{s(0\to1)} \cdot \exp\left(-\lambda_{s(0\to1)} \cdot t\right) \cdot \left(t \cdot M_{(1,0)} + M_{(1,1)} \cdot \left(h_q - t\right)\right) dt \end{aligned} \tag{4.16}$$

$$\begin{aligned} K_{(0,1)}\left(h_q\right) &= M_{(0,1)} \cdot h_q \cdot \exp\left(-\lambda_{x(0\to1)} \cdot h_q\right) + \\ &+ \int_0^{h_q} \lambda_{x(0\to1)} \cdot \exp\left(-\lambda_{x(0\to1)} \cdot t\right) \cdot \left(t \cdot M_{(0,1)} + M_{(1,1)} \cdot \left(h_q - t\right)\right) dt \end{aligned} \tag{4.17}$$

Finally, in the simpler case where state $(1,1)$ is the initial state of a sampling interval of duration h_q, the process will operate under the effect of state $(1,1)$ for the whole interval, and so, the *OOC* operation cost will be equal to:

$$K_{(1,1)}\left(h_q\right) = M_{(1,1)} \cdot h_q \tag{4.18}$$

By utilizing equations (4.12)-(4.18) and after some simple mathematical manipulation, the expected cost and duration of a transition step associated with each of its 12 possible states are summarized in Table 4-1.

Table 4-1: Expected *OOC* operation cost and duration of each transition step of the VP_1 control scheme *

Initial State and Steady-state Probability	Expected *OOC* Operation Cost	Duration
$(Y_t,a_t)=((0,0),0\ or 1)$ $\pi_{(0,0)0\ or 1}$	$K_{(0,0)}(h_q)=M_{(1,0)}\cdot\left(\frac{\lambda_{x(0\to1)}\cdot(1-\exp(-\lambda_{s(0\to1)}\cdot h_q))-\lambda_{s(0\to1)}\cdot\exp(-\lambda_{s(0\to1)}\cdot h_q)\cdot(1-\exp(-\lambda_{x(0\to1)}\cdot h_q))}{\lambda_{s(0\to1)}\cdot(\lambda_{x(0\to1)}+\lambda_{s(0\to1)})}\right)+$ $+M_{(0,1)}\cdot\left(\frac{\lambda_{s(0\to1)}\cdot(1-\exp(-\lambda_{x(0\to1)}\cdot h_q))-\lambda_{x(0\to1)}\cdot\exp(-\lambda_{x(0\to1)}\cdot h_q)\cdot(1-\exp(-\lambda_{s(0\to1)}\cdot h_q))}{\lambda_{x(0\to1)}\cdot(\lambda_{s(0\to1)}+\lambda_{x(0\to1)})}\right)+$ $+M_{(1,1)}\cdot\left(h_q-\frac{(1-\exp(-\lambda_{x(0\to1)}\cdot h_q))}{\lambda_{x(0\to1)}}-\frac{(1-\exp(-\lambda_{s(0\to1)}\cdot h_q))}{\lambda_{s(0\to1)}}+\frac{(1-\exp(-(\lambda_{x(0\to1)}+\lambda_{s(0\to1)})\cdot h_q))}{\lambda_{x(0\to1)}+\lambda_{s(0\to1)}}\right)$	h_q
$(Y_t,a_t)=((0,0),2)$ $\pi_{(0,0)2}$	$K_{(0,0)}(h_q)+L_{(0,0)}$	$h_1+T_{(0,0)}$
$(Y_t,a_t)=((0,1),0\ or 1)$ $\pi_{(0,1)0\ or 1}$	$K_{(0,1)}(h_q)=M_{(0,1)}\cdot\left(\frac{1-\exp(-\lambda_{x(0\to1)}\cdot h_q)}{\lambda_{x(0\to1)}}\right)+M_{(1,1)}\cdot\left(h_q-\frac{1-\exp(-\lambda_{x(0\to1)}\cdot h_q)}{\lambda_{x(0\to1)}}\right)$	h_q
$(Y_t,a_t)=((0,1),2)$ $\pi_{(0,1)2}$	$K_{(0,0)}(h_q)+L_{(0,1)}$	$h_1+T_{(0,1)}$
$(Y_t,a_t)=((1,0),0\ or 1)$ $\pi_{(1,0)0\ or 1}$	$K_{(1,0)}(h_q)=M_{(1,0)}\cdot\left(\frac{1-\exp(-\lambda_{s(0\to1)}\cdot h_q)}{\lambda_{s(0\to1)}}\right)+M_{(1,1)}\cdot\left(h_q-\frac{1-\exp(-\lambda_{s(0\to1)}\cdot h_q)}{\lambda_{s(0\to1)}}\right)$	h_q
$(Y_t,a_t)=((1,0),2)$ $\pi_{(1,0)2}$	$K_{(0,0)}(h_q)+L_{(1,0)}$	$h_1+T_{(1,0)}$
$(Y_t,a_t)=((1,1),0 or 1)$ $\pi_{(1,1)0\ or 1}$	$M_{(1,1)}\cdot h_q$	h_q
$(Y_t,a_t)=((1,1),2)$ $\pi_{(1,1)2}$	$K_{(0,0)}(h_q)+L_{(1,1)}$	$h_1+T_{(1,1)}$

* q=1(2) for $a_{t-1}=0(1)$

Thus, the expected cost of a transition step, *EC*, is derived by multiplying the expected cost associated with each of the 12 possible initial states with the respective steady-state probability and is derived from the following equation:

$$EC = b + \sum_{k=0}^{1}\sum_{l=0}^{1}\pi_{(k,l)0}\cdot\left[c\cdot n_1 + K_{(k,l)}\left(h_1\right)\right] + \sum_{k=0}^{1}\sum_{l=0}^{1}\pi_{(k,l)1}\cdot\left[c\cdot n_2 + K_{(k,l)}\left(h_2\right)\right] + \\ + \sum_{k=0}^{1}\sum_{l=0}^{1}\pi_{(k,l)2}\cdot\left[c\cdot n_1 + K_{(0,0)}\left(h_1\right) + L_{(k,l)}\right] \tag{4.19}$$

In a similar manner, the average duration of a transition step, *ET*, is the weighted average of the durations associated with each steady-state of the Markov chain. It can be computed as the sum of the relaxed (h_1) or tightened (h_2) duration of a sampling interval, plus the time to search and remove an/some assignable cause/es, if needed, multiplied by the respective long-run probabilities for each decision:

$$ET = h_1\cdot\sum_{k=0}^{1}\sum_{l=0}^{1}\pi_{(k,l)0} + h_2\cdot\sum_{k=0}^{1}\sum_{l=0}^{1}\pi_{(k,l)1} + \sum_{k=0}^{1}\sum_{l=0}^{1}\pi_{(k,l)2}\cdot\left(h_1 + T_{(k,l)}\right) \tag{4.20}$$

It should be noted that the value of h_2 is also allowed to be equal to zero. In other words, if the tightened parameters are to be used, the next sample $\left(n_2\right)$ is allowed to be collected immediately after the previous one without allowing the process to operate in between $\left(h_2 = 0.0\right)$. Thus, a sample n_2 is allowed to be collected again (and again) until a signal is issued or until the chart indicates that the relaxed parameters should be used thereafter. Such sampling policy, which is more applicable and feasible in cases where production ceases during the sampling procedure, resembles the double sampling policy as proposed in many papers (Croasdale, 1974, Daudin, 1992). The only difference in this case is that the fixed sampling cost, *b*, should not be added to *EC*, because, the process does not need to be interrupted again. Particularly, the form of *EC*, if $h_2 = 0.0$, is simplified to the following:

$$EC = b \cdot \left(\sum_{k=0}^{1} \sum_{l=0}^{1} \pi_{(k,l)0} + \sum_{k=0}^{1} \sum_{l=0}^{1} \pi_{(k,l)2} \right) + \sum_{k=0}^{1} \sum_{l=0}^{1} \pi_{(k,l)0} \cdot \left[c \cdot n_1 + K_{(k,l)}\left(h_1\right) \right] +$$
$$+ \sum_{k=0}^{1} \sum_{l=0}^{1} \pi_{(k,l)1} \cdot \left[c \cdot n_2 + K_{(k,l)}\left(h_2\right) \right] + \tag{4.21}$$
$$+ \sum_{k=0}^{1} \sum_{l=0}^{1} \pi_{(k,l)2} \cdot \left[c \cdot n_1 + K_{(0,0)}\left(h_1\right) + L_{(k,l)} \right]$$

It is interesting to note that the consideration of the value $h_2 = 0.0$ leads to the possibility of consecutive and repetitive inspection which may seem a bit awkward, or even unrealistic, since the usual assumption in typical *SPC* procedures is that some time must elapse between the collection of two successive samples. However, in this chapter we have assumed negligible times to collect and measure the samples and thus, we can assume that the whole process can be considered to be simultaneous, an assumption not at all unrealistic in modern processes where such complex schemes are expected to be implemented. Additionally, if these assumptions are far from realistic, then, the choice of a minimum value for the sampling interval becomes problematic. The reason is that in such cases, one should also make sure that the inspection rate is high enough to assure that a decision will have been made prior to the collection of the next sample. Moreover, it should also be examined if the interval between two successive sampling instances is large enough to assure that the number of products that will have been produced is at least equal (or larger) than the size of the sample. Such approaches, which are discussed in Celano *et al*. (2011), are beyond the scope of the present thesis, where we assume that the inspection and production rates are high, while the necessary time to measure the mean and standard deviation of the sample, and then reach a decision, is negligible.

This type of inspection policy broadens the applicability of the proposed model, since it allows (but not imposes) a “multiple sampling” feature which may lead to economic savings. That is, allowing h_2 to be equal to zero, may lead to relatively small values of n_2, since the model allows the sampling procedure to continue without charging it again (and again) with the fixed sampling cost b. This assumption does not affect in any way the model development and leaves it to the optimization procedure to “decide” whether the optimum value of h_2 will be zero or larger.

Despite all the above, in the numerical investigation section that follows, we also discuss the choice of setting a lower bound in the allowable values of h_2 and we provide some indicative results when h_2 is inferiorly bounded by a minimum value (say 0.1).

4.4 Optimization Problem

The optimization problem is formulated as follows:

$$
\begin{aligned}
& \min_{DP_q} ECT \\
\text{s.t.}\quad & h_1, h_2, n_1, n_2, w_x, k_{x,1}, k_{x,2}, w_s, k_{s,1}, k_{s,2} > 0 \\
& h_2 \le h_1 \\
& n_2 \ge n_1 \\
& w_x \le k_{x,2} \le k_{x,1} \\
& w_s \le k_{s,2} \le k_{s,1} \\
& n_1, n_2 \in \mathbb{N}^{+}
\end{aligned}
\tag{4.22}
$$

The minimization of *ECT* is achieved by means of a computer program developed in Fortran Power Station 4.0, which estimates the minimum *ECT* and defines the optimum design parameters of the control scheme.

4.5 An Illustrative Example

In order to explain better how the proposed VP_1 scheme is used in practice, a numerical example is utilized. Let us consider a case where $\lambda = 0.005$, $\delta_1 = 0.5$, $\gamma_1^2 = 2.0$ and assume $\mu_0 = 100$ and $\sigma_0 = 10$. Furthermore, as regards the cost elements, $b = 0$, $c = 1$, $L_{(0,0)} = 100$, $L_{(0,1)} = L_{(1,0)} = 200$, $L_{(1,1)} = 300$, $M_{(0,1)} = M_{(1,0)} = 100$ and $M_{(1,1)} = 150$ and the time needed to reveal a false alarm and the time to search and remove an assignable cause are assumed to be negligible.

The optimum scheme's operation employs two allowable values for the sampling interval, $h_1 = 4.0$ and $h_2 = 0.0$, two allowable values for the sample size, $n_1 = 7$ and $n_2 = 16$, warning limit coefficients $w_x = 0.8$ and $w_s = 1.1$ and control limit coefficients $k_{x,1} = 2.6$, $k_{x,2} = 2.0$ for the chart that monitors the process mean and $k_{s,1} = 2.8$, $k_{s,2} = 2.2$ for the chart utilized for monitoring the standard deviation of the process. Consequently, the warning and control limits of the two charts are shown in Figure 4-3.

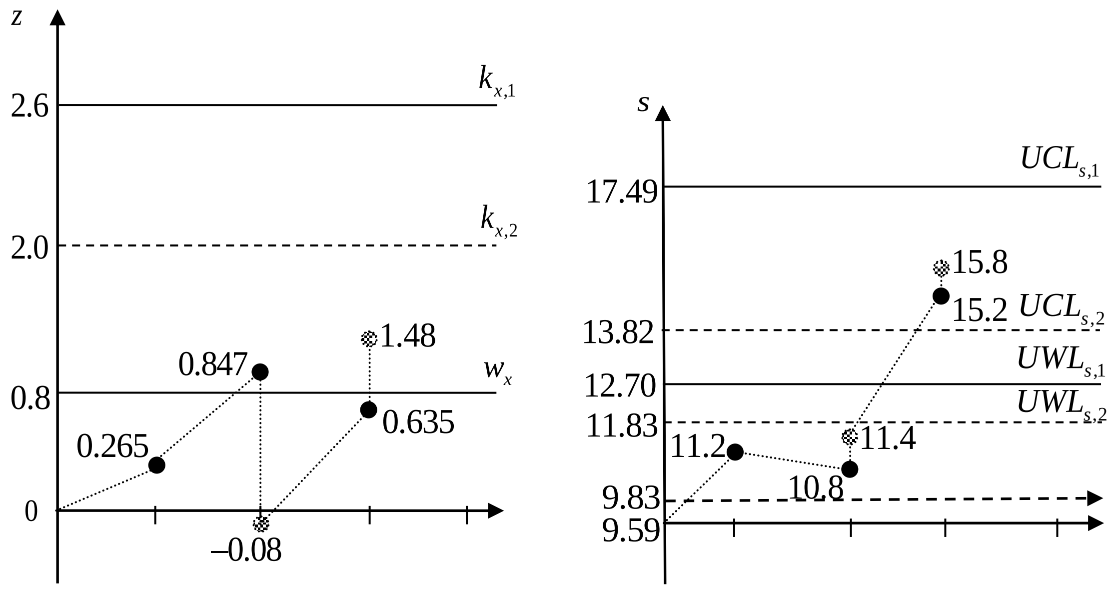


Figure 4-3: Example of the VP_1 control scheme operation

As shown in Figure 4-3, and assuming that the process starts *IC*, the first sample will be collected $h = h_1 = 4.0$ time units after the beginning of the run (or after an intervention to the process). The sample size is 7 $(n_1 = 7)$. Let the first sample mean be $\overline{X} = 101$ and so, $z_1 = \dfrac{101-100}{10/\sqrt{7}} = 0.265$ $\left(= \dfrac{\overline{X} - \mu_0}{\sigma_0 / \sqrt{n_1}} \right)$ and the first sample standard deviation be $s_1 = 11.2$. Since both statistics fall in the central zone, the process is left to operate for another $h = h_1 = 4.0$ time units. After that time (8.0 time units from the beginning), a new sample $n = n_1 = 7$ is collected and its mean is recorded. Let $\overline{X} = 103.2$ $(z_2 = 0.847)$ and $s_2 = 10.8$; we see that although *s* lies again in the central zone, the value of the standardized mean is above the warning limit $(w_x = 0.8)$ and thus, a new sample of $n = n_2 = 16$ units is immediately collected (we represent the outcomes of the tightened inspection –with $n = n_2 = 16$– with a spotted dot to facilitate the illustration of the comparisons that now need to be made with the tightened limits). Let the observation of the new sample be $\overline{X} = 99.8$ and so $z_3 = \frac{99.8 - 100}{10/\sqrt{16}} = -0.08$ $\left(= \frac{\overline{X} - \mu_0}{\sigma_0 / \sqrt{n_2}} \right)$ and $s_3 = 11.4$. Thus, both statistics lie again in the central zones (inside the dotted lines) and thus, after that the process is left again to operate for another $h = h_1 = 4.0$ time units and the next sample $(n = n_1 = 7)$ be collected, 12.0 time units after the beginning of the operation. Let us assume that the second assignable cause occurs at some time between $t = 8.0$ and $t = 12.0$ time units. Let also the new sample outcome be $\overline{X} = 102.4$ $(z_4 = 0.635)$ and $s_4 = 15.2$. Since we are still in a relaxed inspection, we see that the value of s falls in the warning zone. A new sample of size $n = n_2 = 16$ is thus, immediately collected which is assumed to be $\overline{X} = 103.7$ $(z_5 = 1.48)$ and $s_5 = 15.8$. The s chart at this point reveals the occurrence of the assignable cause since the statistic is above the tightened control limit. An investigation follows which reveals the occurrence of the specific assignable cause, the effect of which is immediately removed before the process is allowed to resume its operation again. After the removal of the cause, the next sample will be scheduled to be collected after another $h = h_1 = 4.0$ time units and its size will be $n = n_1 = 7$.

4.6 Numerical Analysis

In this section, the aforementioned approach for computing the optimum design parameters of the control scheme and the minimum expected cost per time unit is applied to 64 cases, with different statistical and economic parameters. The benchmark of process scenarios that are used in the numerical investigation section are presented in Table 4-2. Each case is defined by the value of the cost elements: c, b, $L_{(0,0)}$, $L_{(i,j)}$, $M_{(i,j)}$, the times $T_{(0,0)}$, $T_{(i,j)}$ and the value of the statistical parameters: λ, δ, and γ. It should be mentioned that $\delta_1 = \delta$, $\gamma_1 = \gamma$, $\lambda_{x(0\to1)} = \lambda_{s(0\to1)} = \lambda$ and $M_{(0,1)} = M_{(1,0)} = M$, $M_{(1,1)} = 1.5M$. Moreover, for all cases: $c = 1$, $L_{(0,1)} = L_{(1,0)} = 200$, $L_{(1,1)} = 300$, $T_{(0,0)} = T_{(0,1)} = T_{(1,0)} = T_{(1,1)} = 0$.

Table 4-2: Parameter sets of the 64 numerical examples for the VP_1 control scheme

Case	b	M	$L_{(0,0)}$	λ	δ	γ	Case	b	M	$L_{(0,0)}$	λ	δ	γ
1	0	100	100	0.005	0.5	1.414	33	0	100	100	0.005	0.5	2.0
2	0	100	200	0.005	0.5	1.414	34	0	100	200	0.005	0.5	2.0
3	0	1000	100	0.005	0.5	1.414	35	0	1000	100	0.005	0.5	2.0
4	0	1000	200	0.005	0.5	1.414	36	0	1000	200	0.005	0.5	2.0
5	5	100	100	0.005	0.5	1.414	37	5	100	100	0.005	0.5	2.0
6	5	100	200	0.005	0.5	1.414	38	5	100	200	0.005	0.5	2.0
7	5	1000	100	0.005	0.5	1.414	39	5	1000	100	0.005	0.5	2.0
8	5	1000	200	0.005	0.5	1.414	40	5	1000	200	0.005	0.5	2.0
9	0	100	100	0.05	0.5	1.414	41	0	100	100	0.05	0.5	2.0
10	0	100	200	0.05	0.5	1.414	42	0	100	200	0.05	0.5	2.0
11	0	1000	100	0.05	0.5	1.414	43	0	1000	100	0.05	0.5	2.0
12	0	1000	200	0.05	0.5	1.414	44	0	1000	200	0.05	0.5	2.0
13	5	100	100	0.05	0.5	1.414	45	5	100	100	0.05	0.5	2.0
14	5	100	200	0.05	0.5	1.414	46	5	100	200	0.05	0.5	2.0
15	5	1000	100	0.05	0.5	1.414	47	5	1000	100	0.05	0.5	2.0
16	5	1000	200	0.05	0.5	1.414	48	5	1000	200	0.05	0.5	2.0
17	0	100	100	0.005	1.0	1.414	49	0	100	100	0.005	1.0	2.0
18	0	100	200	0.005	1.0	1.414	50	0	100	200	0.005	1.0	2.0
19	0	1000	100	0.005	1.0	1.414	51	0	1000	100	0.005	1.0	2.0
20	0	1000	200	0.005	1.0	1.414	52	0	1000	200	0.005	1.0	2.0
21	5	100	100	0.005	1.0	1.414	53	5	100	100	0.005	1.0	2.0
22	5	100	200	0.005	1.0	1.414	54	5	100	200	0.005	1.0	2.0
23	5	1000	100	0.005	1.0	1.414	55	5	1000	100	0.005	1.0	2.0
24	5	1000	200	0.005	1.0	1.414	56	5	1000	200	0.005	1.0	2.0
25	0	100	100	0.05	1.0	1.414	57	0	100	100	0.05	1.0	2.0
26	0	100	200	0.05	1.0	1.414	58	0	100	200	0.05	1.0	2.0
27	0	1000	100	0.05	1.0	1.414	59	0	1000	100	0.05	1.0	2.0
28	0	1000	200	0.05	1.0	1.414	60	0	1000	200	0.05	1.0	2.0
29	5	100	100	0.05	1.0	1.414	61	5	100	100	0.05	1.0	2.0
30	5	100	200	0.05	1.0	1.414	62	5	100	200	0.05	1.0	2.0
31	5	1000	100	0.05	1.0	1.414	63	5	1000	100	0.05	1.0	2.0
32	5	1000	200	0.05	1.0	1.414	64	5	1000	200	0.05	1.0	2.0

The economic outcome of the proposed VP_1 control scheme for each case as well as the optimum parameters are presented in Tables 4-3 and 4-4.

Table 4-3: Economic design for numerical examples 1-32: optimal control policy and cost for the VP_1 control scheme

Optimum Design Parameters												*Statistical Measures*						
Case	h_1	h_2	n_1	n_2	w_x	$k_{x,1}$	$k_{x,2}$	w_s	$k_{s,1}$	$k_{s,2}$	ECT_{VP1}	α	$1\text{-}\beta$	*ANOF*	ARL_0	*WARL*	*ATC*	*EATR*
1	4.0	0.0	7	16	0.8	2.6	2.0	1.1	2.8	2.2	10.56	0.0192	0.3266	0.0006	52.19	3.06	114.56	4.56
2	4.4	0.0	9	24	1.0	3.1	2.3	1.2	3.3	2.5	10.89	0.0065	0.3318	0.0002	152.88	3.01	104.67	4.67
3	1.2	0.0	7	18	0.9	2.8	2.0	1.1	3.0	2.3	29.45	0.0145	0.3199	0.0006	68.82	3.13	101.41	1.41
4	1.3	0.0	9	24	1.0	3.3	2.4	1.2	3.4	2.6	30.30	0.0049	0.3135	0.0002	205.83	3.19	101.41	1.41
5	6.5	0.0	13	18	0.8	2.3	1.9	1.0	2.4	2.2	11.47	0.0296	0.4620	0.0005	33.78	2.165	105.09	5.09
6	6.7	0.0	14	25	0.9	2.8	2.3	1.1	2.9	2.5	11.79	0.0097	0.4119	0.0002	103.28	2.43	105.23	5.23
7	2.0	0.0	13	19	0.8	2.4	2.0	1.0	2.5	2.3	32.54	0.0234	0.4401	0.0005	42.72	2.27	101.57	1.57
8	2.0	0.0	14	27	1.0	3.0	2.4	1.1	3.1	2.6	33.30	0.0063	0.3906	0.0001	157.58	2.56	101.60	1.60
9	1.7	0.0	8	13	0.8	2.2	1.7	1.0	2.4	2.1	43.95	0.0386	0.3896	0.0006	25.93	2.57	11.75	1.75
10	1.9	0.0	10	19	0.9	2.6	2.0	1.1	2.8	2.3	45.45	0.0170	0.3830	0.0002	58.86	2.61	11.82	1.82
11	0.4	0.0	7	16	0.8	2.6	2.0	1.1	2.8	2.2	105.55	0.0192	0.3266	0.0006	52.19	3.06	10.46	0.46
12	0.4	0.0	8	23	1.0	3.2	2.3	1.2	3.3	2.5	108.98	0.0064	0.3146	0.0002	156.01	3.18	10.46	0.46
13	2.6	0.0	13	14	0.7	1.9	1.8	0.9	2.1	2.1	46.21	0.0531	0.5229	0.0005	18.83	1.91	11.96	1.96
14	2.7	0.0	15	20	0.9	2.3	2.0	1.0	2.5	2.3	47.59	0.0245	0.4912	0.0002	40.78	2.04	12.03	2.03
15	0.7	0.0	14	19	0.8	2.2	2.0	1.0	2.4	2.2	114.78	0.0298	0.4842	0.0005	33.52	2.07	10.52	0.52
16	0.7	0.0	15	25	0.9	2.8	2.3	1.1	2.9	2.5	117.93	0.0096	0.4208	0.0002	103.74	2.38	10.53	0.53
17	3.1	0.0	5	12	1.4	3.1	2.8	0.9	3.0	2.0	8.83	0.0011	0.3387	0.0005	90.49	2.95	103.64	3.64
18	3.0	0.0	5	16	1.5	3.5	3.0	1.0	3.6	2.3	9.09	0.0042	0.3198	0.0002	238.85	3.13	103.67	3.67
19	0.9	0.0	5	12	1.4	3.2	2.9	0.9	3.1	2.1	23.89	0.0088	0.3209	0.0005	113.22	3.12	101.09	1.09
20	1.0	0.0	6	18	1.6	3.7	3.0	1.0	3.6	2.4	24.60	0.0032	0.3360	0.0002	310.24	2.98	101.12	1.12
21	5.5	0.0	9	14	1.5	2.9	2.9	0.7	2.5	2.0	9.98	0.0167	0.4801	0.0004	59.74	2.08	104.27	4.27
22	5.5	0.0	9	18	1.5	3.2	3.0	0.8	3.1	2.3	10.20	0.0067	0.4341	0.0001	161.90	2.30	104.38	4.38
23	1.7	0.0	9	15	1.5	3.1	3.0	0.7	2.6	2.1	27.68	0.0013	0.4572	0.0004	76.77	2.19	101.32	1.32
24	1.7	0.0	9	20	1.6	3.5	3.0	0.8	3.2	2.4	28.27	0.0046	0.4203	0.0001	218.90	2.38	101.35	1.35
25	1.2	0.0	5	10	1.3	2.7	2.5	0.9	2.7	1.9	39.37	0.0185	0.3718	0.0004	53.93	2.69	11.36	1.36
26	1.3	0.0	6	13	1.4	3.0	2.8	0.9	3.1	2.2	40.40	0.0082	0.3631	0.0002	122.39	2.75	11.37	1.37
27	0.3	0.0	5	12	1.4	3.1	2.8	0.9	3.0	2.0	88.31	0.0011	0.3386	0.0005	90.49	2.95	10.35	0.35
28	0.3	0.0	5	16	1.5	3.5	3.0	1.0	3.6	2.3	90.92	0.0042	0.3198	0.0002	238.85	3.13	10.37	0.37
29	2.1	0.0	8	11	1.3	2.5	2.5	0.6	2.3	1.9	42.37	0.0280	0.5072	0.0004	35.72	1.97	11.63	1.63
30	2.1	0.0	9	15	1.5	2.8	2.8	0.7	2.8	2.2	43.25	0.0111	0.4736	0.0001	89.97	2.11	11.64	1.64
31	0.5	0.0	8	13	1.4	2.9	2.8	0.7	2.6	2.0	99.92	0.0159	0.4461	0.0004	62.94	2.24	10.41	0.41
32	0.6	0.0	10	19	1.6	3.2	3.0	0.8	3.0	2.3	102.13	0.0065	0.4618	0.0001	154.66	2.17	10.46	0.46

Table 4-4: Economic design for numerical examples 33-64: optimal control policy and cost for the VP_1 control scheme

Optimum Design Parameters												*Statistical Measures*						
Case	h_1	h_2	n_1	n_2	w_x	$k_{x,1}$	$k_{x,2}$	w_s	$k_{s,1}$	$k_{s,2}$	ECT_{VP1}	α	$1\text{-}\beta$	*ANOF*	ARL_0	*WARL*	*ATC*	*EATR*
33	3.3	0.0	5	13	0.7	3.0	1.9	1.8	3.8	3.5	8.82	0.0092	0.3400	0.0004	108.76	2.94	103.57	3.57
34	3.7	0.0	6	17	0.8	3.5	2.1	1.9	4.1	3.7	9.06	0.0046	0.3512	0.0002	217.08	2.85	103.73	3.73
35	1.0	0.0	5	13	0.7	3.1	1.9	1.8	4.0	3.6	23.81	0.0087	0.3323	0.0005	114.58	3.01	101.09	1.09
36	1.1	0.0	6	17	0.7	3.6	2.2	1.9	4.3	3.9	24.43	0.0039	0.3400	0.0002	254.63	2.94	101.05	1.05
37	5.7	0.0	9	15	0.6	2.5	1.9	1.8	3.4	3.4	9.92	0.0137	0.4643	0.0003	72.87	2.15	104.21	4.21
38	5.7	0.0	9	19	0.7	3.1	2.1	1.9	3.8	3.8	10.12	0.0057	0.4292	0.0001	175.43	2.33	104.32	4.32
39	1.7	0.0	8	15	0.6	2.7	1.9	1.7	3.7	3.7	27.44	0.0115	0.4249	0.0003	87.18	2.35	101.31	1.31
40	1.7	0.0	9	19	0.7	3.3	2.2	1.9	4.0	4.0	27.93	0.0042	0.4187	0.0001	238.81	2.48	101.30	1.30
41	1.3	0.0	5	10	0.6	2.6	1.7	1.7	3.3	3.2	39.50	0.0182	0.3671	0.0004	54.82	2.72	11.35	1.35
42	1.4	0.0	6	15	0.8	3.0	1.9	0.8	3.6	3.4	40.45	0.0121	0.3987	0.0003	82.72	2.51	11.18	1.18
43	0.3	0.0	5	13	0.8	2.7	1.8	1.9	3.8	3.4	88.48	0.0115	0.3504	0.0005	86.82	2.85	10.35	0.35
44	0.4	0.0	7	17	0.8	3.3	2.1	1.9	4.0	3.7	90.70	0.0048	0.3686	0.0002	207.94	2.71	10.37	0.37
45	2.1	0.0	8	11	0.5	2.2	1.7	1.6	3.0	3.0	42.34	0.0268	0.4853	0.0003	37.29	2.06	11.59	1.59
46	2.2	0.0	9	15	0.6	2.7	1.9	1.8	3.3	3.3	43.19	0.0121	0.4661	0.0001	82.42	2.15	11.64	1.64
47	0.6	0.0	9	15	0.6	2.5	1.8	1.7	3.4	3.4	99.29	0.0159	0.4777	0.0004	62.71	2.09	10.44	0.44
48	0.6	0.0	10	19	0.7	3.0	2.1	1.9	3.7	3.7	101.24	0.0061	0.4479	0.0001	65.10	2.23	10.44	0.44
49	2.8	0.0	4	9	1.2	3.3	2.6	1.5	3.8	3.1	6.67	0.0022	0.4032	0.0001	460.02	2.48	102.45	2.45
50	2.8	0.0	4	9	1.2	3.6	2.8	1.5	4.1	3.3	6.73	0.0011	0.3695	0.0001	933.56	2.71	102.49	2.49
51	0.7	0.0	3	8	1.1	3.6	2.7	1.5	3.9	3.2	16.75	0.0018	0.3393	0.0002	562.79	2.95	100.72	0.72
52	0.7	0.0	3	9	1.2	3.9	2.9	1.6	4.5	3.4	16.90	0.0007	0.3214	0.0001	1494.15	3.11	100.76	0.76
53	4.8	0.0	6	9	1.1	3.0	2.6	1.3	3.4	3.1	8.01	0.0036	0.5013	0.0001	278.53	1.99	103.18	3.18
54	4.8	0.0	6	10	1.1	3.4	2.8	1.3	3.8	3.3	8.06	0.0015	0.4591	0.0001	688.78	2.18	103.16	3.16
55	1.5	0.0	6	10	1.1	3.2	2.8	1.3	3.6	3.2	21.19	0.0021	0.4782	0.0001	485.18	2.09	100.98	0.98
56	1.5	0.0	6	10	1.1	3.5	2.9	1.3	3.8	3.4	21.31	0.0011	0.4419	0.0001	871.76	2.26	100.99	0.99
57	0.8	0.0	3	7	1.1	3.1	2.4	1.5	3.6	3.0	33.70	0.0043	0.3661	0.0001	234.79	2.73	10.83	0.83
58	1.0	0.0	4	9	1.2	3.4	2.6	1.5	3.9	3.2	34.04	0.0018	0.3991	0.0001	543.81	2.51	10.88	0.88
59	0.2	0.0	3	9	1.3	3.5	2.6	1.7	4.2	3.1	66.72	0.0015	0.3484	0.0001	658.99	2.87	10.23	0.23
60	0.2	0.0	3	9	1.3	3.8	2.8	1.7	4.5	3.3	67.31	0.0008	0.3241	0.0001	1294.84	3.09	10.23	0.23
61	1.7	0.0	6	8	1.1	2.7	2.4	1.2	3.2	2.9	37.37	0.0071	0.5449	0.0001	139.94	1.84	11.13	1.13
62	1.7	0.0	6	9	1.1	3.0	2.6	1.3	3.5	3.2	37.65	0.0032	0.5006	0.0001	308.79	2.00	11.14	1.14
63	0.5	0.0	6	9	1.1	3.0	2.6	1.2	3.4	3.1	80.14	0.0037	0.5023	0.0001	273.73	1.99	10.33	0.33
64	0.5	0.0	6	10	1.1	3.4	2.8	1.3	3.7	3.3	80.69	0.0016	0.4624	0.0001	641.38	2.16	10.33	0.33

By examining the optimum design parameters of the VP_1 control scheme, it is immediately evident that the value of the sampling interval for the tightened sampling, namely h_2, is equal to zero, in all 64 cases. As mentioned earlier, if $h_2 = 0$, the next sampling should be taken immediately after the previous one.

Another conclusion that can be derived from the examination of the optimum design parameters of the VP_1 control scheme is that, for large $M_{(i,j)}$ costs, the optimum value of h_1 gets smaller and *ECT* increases. This indicates that if the operation cost in presence of an assignable cause is great, sampling intervals should be smaller, so as to avoid a long-lasting *OOC* process operation. The same conclusions, namely greater *ECT* and smaller sampling intervals, can be drawn for larger values of $\lambda's$. Moreover, if the shifts δ_1 and γ_1 of the mean and standard deviation, respectively, are large, then the minimum *ECT* gets smaller because it becomes easier to detect the assignable causes: the warning and control limits of both control charts get larger and sampling intervals get smaller.

Furthermore, as regards the statistical measures, the *Type I* error probability a $\left(ARL_0\right)$ is positively (inversely) correlated to the occurrence rates and inversely (positively) correlated to the effects of the assignable causes on the process. Additionally, *ANOF* is similarly to a affected by λ, δ and γ. Moreover, *ATC* and *EATR* are greater in case of lower occurrence rates.

The comparison of the economic outcome of the proposed VP_1 control scheme with other control schemes that monitor both the process mean and the standard deviation, but have fewer adaptive parameters, is given in Tables 4-5 and 4-6. In particular, the VP_1 control scheme is compared to the respective one-sided static control scheme, namely *FP* control scheme, and to the following one-sided adaptive control charts: (a) *VSS* control scheme; (b) *VSI* control scheme; (c) *VSSI* control scheme.

Table 4-5: Economic comparison between the VP_1 control scheme and other less adaptive control schemes. Numerical examples 1-32

	ECT					$\frac{FP-VP_1}{FP}$	$\frac{VSS-VP_1}{VSS}$	$\frac{VSI-VP_1}{VSI}$	$\frac{VSSI-VP_1}{VSSI}$
Case	*FP*	*VSS*	*VSI*	*VSSI*	VP_1	(%)	(%)	(%)	(%)
1	11.59	11.50	10.72	10.62	10.56	8.89	8.17	1.49	0.56
2	12.54	12.48	11.27	11.02	10.89	13.16	12.74	3.37	1.18
3	32.25	32.97	30.22	29.71	29.45	8.68	10.68	2.55	0.88
4	36.36	36.15	31.79	30.70	30.30	16.67	16.18	4.69	1.30
5	12.29	12.27	11.52	11.48	11.47	6.67	6.52	0.43	0.09
6	13.17	13.24	11.97	11.83	11.79	10.48	10.95	1.50	0.34
7	35.54	35.48	32.79	32.60	32.54	8.44	8.29	0.76	0.18
8	38.38	38.65	34.14	33.46	33.30	13.24	13.84	2.46	0.48
9	45.86	45.70	44.09	44.02	43.95	4.16	3.83	0.32	0.16
10	48.43	48.28	45.85	45.65	45.45	6.15	5.86	0.87	0.44
11	115.89	115.06	107.33	106.24	105.55	8.92	8.27	1.66	0.65
12	125.44	124.75	112.65	110.17	108.98	13.12	12.64	3.26	1.08
13	47.81	47.75	46.22	46.22	46.21	3.35	3.23	0.02	0.02
14	50.15	50.21	47.73	47.65	47.59	5.10	5.22	0.29	0.13
15	122.95	122.70	115.17	114.87	114.78	6.64	6.45	0.34	0.08
16	131.70	132.37	119.80	118.34	117.93	10.46	10.91	1.56	0.35
17	9.92	9.81	9.08	8.94	8.83	10.99	9.99	2.75	1.23
18	10.70	10.49	9.52	9.26	9.09	15.05	13.35	4.52	1.84
19	27.77	27.36	24.83	24.26	23.89	13.97	12.68	3.79	1.53
20	30.31	29.52	26.21	25.20	24.60	18.84	16.67	6.14	2.38
21	10.84	10.78	10.07	10.02	9.98	7.93	7.42	0.89	0.40
22	11.53	11.42	10.42	10.28	10.20	11.54	10.68	2.11	0.78
23	30.74	30.53	28.06	27.81	27.68	9.95	9.34	1.35	0.47
24	33.01	32.58	29.15	28.54	28.27	14.36	13.23	3.02	0.95
25	41.61	41.41	39.74	39.51	39.37	5.38	4.93	0.93	0.35
26	43.62	43.30	41.09	40.78	40.40	7.38	6.70	1.68	0.93
27	99.33	98.12	90.77	89.51	88.31	11.09	10.00	2.71	1.34
28	107.01	104.95	95.17	92.64	90.92	15.04	13.37	4.47	1.86
29	44.15	44.05	42.46	42.42	42.37	4.03	3.81	0.21	0.12
30	45.92	45.74	43.54	43.39	43.25	5.81	5.44	0.67	0.32
31	108.48	107.88	100.74	100.28	99.92	7.89	7.38	0.81	0.36
32	115.28	114.14	104.21	102.79	102.13	11.41	10.52	2.00	0.64

Table 4-6: Economic comparison between the VP_1 control scheme and other less adaptive control schemes. Numerical examples 33-64

Case	ECT: FP	ECT: VSS	ECT: VSI	ECT: $VSSI$	ECT: VP_1	$\frac{FP-VP_1}{FP}$ (%)	$\frac{VSS-VP_1}{VSS}$ (%)	$\frac{VSI-VP_1}{VSI}$ (%)	$\frac{VSSI-VP_1}{VSSI}$ (%)
33	10.04	9.89	9.15	8.98	8.82	12.15	10.82	3.61	1.78
34	10.86	10.61	9.60	9.27	9.06	16.57	14.61	5.63	2.27
35	28.15	27.62	25.05	24.32	23.81	15.42	13.79	4.95	2.10
36	30.86	29.89	26.43	25.05	24.43	20.84	18.27	7.57	2.48
37	10.88	10.82	10.06	9.98	9.92	8.82	8.32	1.39	0.60
38	11.62	11.50	10.43	10.23	10.12	12.91	12.00	2.97	1.08
39	30.85	30.64	27.97	27.64	27.44	11.05	10.44	1.89	0.72
40	33.29	32.90	29.09	28.27	27.93	16.10	15.11	3.99	1.20
41	41.91	41.65	40.02	39.83	39.50	5.75	5.16	1.30	0.83
42	44.05	43.62	41.44	41.00	40.45	8.17	7.27	2.39	1.34
43	100.43	98.98	91.55	89.82	88.48	11.90	10.61	3.35	1.49
44	108.63	106.03	96.16	92.66	90.70	16.51	14.46	5.68	2.12
45	44.25	44.16	42.51	42.45	42.34	4.32	4.12	0.40	0.26
46	46.15	45.96	43.67	43.45	43.19	6.41	6.03	1.10	0.60
47	108.84	108.28	100.6	99.83	99.29	8.77	8.30	1.30	0.54
48	116.21	115.05	104.31	102.30	101.24	12.88	12.00	2.94	1.04
49	7.76	7.68	6.91	6.71	6.67	14.05	13.15	3.47	0.60
50	8.12	8.00	7.06	6.80	6.73	17.12	15.88	4.67	1.03
51	20.55	20.29	17.67	16.93	16.75	18.49	17.45	5.21	1.06
52	21.73	21.17	18.12	17.21	16.90	22.23	20.17	6.73	1.80
53	8.80	8.78	8.07	8.02	8.01	8.98	8.77	0.74	0.12
54	9.09	9.06	8.17	8.08	8.06	11.33	11.04	1.35	0.25
55	23.89	23.82	21.44	21.21	21.19	11.30	11.04	1.17	0.09
56	24.82	24.75	21.73	21.39	21.31	14.14	13.90	1.93	0.37
57	36.26	36.08	34.19	33.86	33.70	7.06	6.60	1.43	0.47
58	37.26	37.07	34.72	34.24	34.04	8.64	8.17	1.96	0.58
59	77.60	76.82	69.18	67.19	66.72	14.02	13.15	3.56	0.70
60	81.34	80.11	70.68	68.04	67.31	17.25	15.98	4.77	1.07
61	39.18	39.16	37.45	37.86	37.37	4.62	4.57	0.21	1.29
62	39.99	39.95	37.82	37.70	37.65	5.85	5.76	0.45	0.13
63	88.05	87.93	80.73	80.24	80.14	8.98	8.86	0.73	0.12
64	90.87	90.66	81.83	80.88	80.69	11.20	11.00	1.39	0.23

The percentage improvements presented in the last columns of the two tables indicate a significant economic improvement of the proposed model, compared to other less sophisticated control schemes. Particularly, the percentage improvement of the economic outcome achieved by the use of VP_1 control scheme, varies between 3.35% and 22.23% when compared to the *FP* control scheme; between 3.23% and 20.17% when compared to the *VSS* control scheme; between 0.02% and 7.57% when compared to the *VSI* control scheme and between 0.02% and 2.48% when compared to the *VSSI* control scheme, with a mean percentage improvement of 10.92%, 10.25%, 2.40% and 0.84%, respectively.

In more detail, for the entire benchmark of cases, the proposed VP_1 control scheme has an improved economic outcome when compared to all other control schemes. The improvement is greater in cases where $M_{(i,j)}$, $L_{(0,0)}$, δ and γ are larger and *b*, λ are smaller. Besides the economic superiority of the VP_1 control scheme compared to other control schemes, the comparison of costs indicates that *VSSI* control scheme has a better economic performance compared to *VSI*, *VSS* and the *FP* control scheme performance. Moreover, the *VSI* control scheme outperforms the *VSS* control scheme in all 64 cases that are examined. In other words, the parameter that significantly affects the cost improvement is the sampling interval. Both *VSI* and *VSS* control schemes have obviously a smaller minimum expected cost per time unit compared to the static *FP* control scheme in all the examined cases.

The above conclusions enhance the general conclusion of literature about better economic performance of control charts that are fully adaptive compared to static control charts or adaptive control charts that allow fewer parameters to vary.

To remove any skepticism regarding the advisability of the choice to allow the tightened sampling interval to also take zero values, we have rerun all 64 cases, setting 0.1 as a minimum allowable value for h_2. We indicatively present the optimum design parameters and expected cost of the first case when $h_2 \geq 0.1$: $h_1 = 4.3$, $h_2 = 0.1$, $n_1 = 8$, $n_2 = 18$, $w_x = 0.9$, $k_{x,1} = 2.6$, $k_{x,2} = 1.9$, $w_s = 1.1$, $k_{s,1} = 2.8$, $k_{s,2} = 2.2$ and $ECT = 10.60$. It is evident that this solution is slightly different to the optimum one when h_2 is allowed to be equal to zero (first line of

Table 4-3), while the cost is less than 0.4% increased. The differences, both in the optimum solutions and in the minimum costs, remain relatively low in all cases with $b=0$, since in these cases the consecutive sampling feature does not save any extra fixed sampling cost. On the other hand, things are different when $b>0$. We indicatively present the results of case 5, which is essentially same to case 1 but with the fixed sampling cost b equal to 5. The results of this case when $h_2 \geq 0.1$ are: $h_1 = 6.9$, $h_2 = 0.1$, $n_1 = 17$, $n_2 = 26$, $k_{x,1} = 2.4$, $k_{x,2} = 2.0$, $w_x = 1.1$, $k_{s,1} = 2.6$, $k_{s,2} = 2.3$, $w_s = 1.2$ and $ECT = 11.80$. We see that the optimum cost in this case (and in all cases with $b>0$) is considerably higher, while the optimum sample sizes are considerably larger. It is evident by these solutions that the cost savings may be rather significant when h_2 is allowed to be equal to zero, especially when $b>0$. Thus, to achieve the maximum economic savings, we have allowed $h_2 = 0.0$, we leave the optimization procedure to dictate the optimum, and we just add a note that even in cases where the samplings are not instantaneous, $h_2 = 0.0$ is an indication to the practitioner that they should collect the new sample immediately after the previous one, without charging an additional fixed cost.

So far, in all 64 scenarios, the charts were economically optimized without taking into account their statistical behaviour. However, it must be said that a scheme that leads to unacceptable statistical performance is extremely difficult to be adapted by the practitioners. For example, a control scheme with relatively high false alarm probabilities leads to unnecessary over-adjustments and destroys the confidence to the control procedure. These phenomena occur very often when designing a model from a purely economic point of view, since the statistical behaviour in these cases is completely disregarded. To avoid extreme cases where the statistical performance of the charts is unacceptable, many researchers have considered the so-called economic-statistical design (Saniga, 1989). For ease of reading, Table 4-7 gathers the values of the *Type I* errors incurred by the economically optimum solutions of the VP_1 control scheme, presented in Tables 4-3 and 4-4, for all 64 scenarios.

Table 4-7: *Type I* Error for the 64 numerical examples of the VP_1 control scheme

Case	α	*Case*	α	*Case*	α	*Case*	α
1	0.0192	17	0.0110	33	0.0092	49	0.0022
2	0.0065	18	0.0042	34	0.0046	50	0.0011
3	0.0145	19	0.0088	35	0.0087	51	0.0018
4	0.0049	20	0.0032	36	0.0039	52	0.0007
5	**0.0296**	21	0.0167	37	0.0137	53	0.0036
6	0.0097	22	0.0062	38	0.0057	54	0.0015
7	**0.0234**	23	0.0130	39	0.0115	55	0.0021
8	0.0063	24	0.0046	40	0.0042	56	0.0011
9	**0.0386**	25	0.0185	41	0.0182	57	0.0043
10	0.0170	26	0.0082	42	0.0121	58	0.0018
11	0.0192	27	0.0110	43	0.0115	59	0.0015
12	0.0064	28	0.0042	44	0.0048	60	0.0008
13	**0.0531**	**29**	**0.0280**	**45**	**0.0268**	61	0.0071
14	**0.0245**	30	0.0111	46	0.0121	62	0.0032
15	**0.0298**	31	0.0159	47	0.0159	63	0.0037
16	0.0096	32	0.0065	48	0.0061	64	0.0016

As shown in Table 4-7, there are only few cases with high False Alarm probabilities. In particular, only in 8 (out of the 64 scenarios) there is a probability of a *Type I* error higher than 0.02. These scenarios are optimized again, but with the addition of a statistical constraint for the *Type I* error: $a \leq 0.02$, and a new Table is introduced (Table 4-8). In essence, the optimization procedure remained the same, but it disregarded all solutions where the average *Type I* error exceeded 0.02.

It is obvious from Table 4-8 that the addition of the *Type I* error statistical constraint has resulted in insignificantly different design parameters and costs, in comparison to the respective cases of Tables 4-3 and 4-4. In particular, the *Type I* error constraint has reduced the False Alarm probabilities without essentially altering the optimum solutions or increasing the expected cost, since in most cases, the *Type I* error was anyway relatively low. Only in case 13 was the minimum cost increased by more than 1%, after the implementation of the statistical constraint, since in that case the *Type I* error was relatively high (0.0531) in the unrestricted run.

Table 4-8: Economic-Statistical design for numerical examples with high *Type I* error: optimal control policy and cost for the VP_1 control scheme

Case	*Statistical constraint*	h_1	h_2	n_1	n_2	w_x	$k_{x,1}$	$k_{x,2}$	w_s	$k_{s,1}$	$k_{s,2}$	ECT_{VP1}	a
5	α	6.5	0	13	18	0.8	2.3	1.9	1.0	2.4	2.2	11.47	0.0296
	$a \le 0.02$	6.2	0	12	19	0.8	2.5	2.0	1.0	2.6	2.4	11.49	0.0200
7	α	2.0	0	13	19	0.8	2.4	2.0	1.0	2.5	2.3	32.54	0.0234
	$a \le 0.02$	2.0	0	13	20	0.8	2.5	2.1	1.0	2.6	2.3	32.54	0.0192
9	α	1.7	0	8	13	0.8	2.2	1.7	1.0	2.4	2.1	43.95	0.0385
	$a \le 0.02$	1.4	0	6	14	0.8	2.7	1.9	1.0	2.8	2.3	44.20	0.0199
13	α	2.6	0	13	14	0.7	1.9	1.8	0.9	2.1	2.1	46.21	0.0531
	$a \le 0.02$	2.4	0	12	18	0.8	2.4	2.1	0.9	2.7	2.3	46.76	0.0200
14	α	2.7	0	15	20	0.9	2.3	2.0	1.0	2.5	2.3	47.59	0.0245
	$a \le 0.02$	2.3	0	15	21	0.9	2.4	2.1	1.0	2.6	2.4	47.61	0.0192
15	α	0.7	0	14	19	0.8	2.2	2.0	1.0	2.4	2.2	114.78	0.0298
	$a \le 0.02$	0.6	0	12	19	0.8	2.5	2.0	1.0	2.6	2.4	114.97	0.0200
29	α	2.1	0	8	11	1.3	2.5	2.5	0.6	2.3	1.9	42.37	0.0280
	$a \le 0.02$	2.0	0	8	11	1.3	2.7	2.6	0.6	2.5	2.0	42.40	0.0199
45	α	2.1	0	8	11	0.5	2.2	1.7	1.6	3.0	3.0	42.34	0.0268
	$a \le 0.02$	2.1	0	8	12	0.5	2.4	1.8	1.6	3.1	3.1	42.38	0.0197

From the above, it becomes obvious that the expected cost is a relatively flat function and the optimum (or near-optimum) solutions can be achieved by the use of many alternative set of design parameters, many of which result in an acceptable statistical behavior without essentially sacrificing the economic performance of the scheme.

5. *VP* $\overline{X}$ CONTROL CHART FOR PROCESSES SUBJECT TO MULTIPLE QUALITY SHIFTS AFFECTING LOCATION (*VP*₂)

5.1 Introduction

In this chapter, the economic-statistical design of a one-sided *VP* control chart is performed by means of a general model considering multiple assignable causes (m) that can contemporarily occur and progressively deteriorate the process mean. Consequently, there are m+1 possible values of $\delta \in \{\delta_0 = 0, \delta_1, \delta_2, ..., \delta_{m-1}, \delta_m\}$ such that $\delta_{i-1} < \delta_i$ for $i = 1,...,m$, where $\delta_0 = 0$ refers to the *IC* state.

This chapter is structured as follows. Section 5.2 presents the proposed Markov chain model. Section 5.3 presents the cost expressions needed to compute the quality control hourly cost. In Section 5.4 the optimization problem is presented and Section 5.5 shows the numerical analysis and performance comparisons.

It should be mentioned that this chapter uses material from Nenes, Tasias and Celano (2015).

5.2 Mathematical Model

The proposed *VP* $\bar{X}$ -Shewhart chart, denoted by VP_2, extends and improves the work presented in Nenes (2011) for the *VP* $\bar{X}$ -Shewhart charts with a single assignable cause having different downward and upward shift sizes.

Under the assumptions that only the process mean may be affected by the occurrence of a quality shift, the design parameters of the VP_2 control chart are reduced to $\{n_q, h_q, w_x, k_{x,q}\}$, where $n_1 \le n_2$, $h_1 \ge h_2$ and $w_x \le k_{x,2} \le k_{x,1}$.

Additionally, $a_t = 0$ when the standardized sample mean is plotted below the upper warning limit coefficient w_x, i.e., $z_t \le w_x$, where z_t is the standardized sample mean at the *t*-th sampling inspection. In this case, the decision is to let the process continue its operation and to use the relaxed sampling $\{n_1, h_1, w_x, k_{x,1}\}$ at the next sampling epoch. Similarly, $a_t = 1$ when the standardized sample mean is plotted within the warning zone, i.e, $w_x < z_t \le k_{x,q}$, with $q = 1, 2$. The decision is again to let the process continue its operation but the parameter values at the next sampling epoch should correspond to the tightened sampling $\{n_2, h_2, w_x, k_{x,2}\}$. Finally, $a_t = 2$ when the chart triggers a signal $(z_t > k_{x,q})$, $q = 1, 2$, and the decision is to stop the process for investigation and possible restoration. After restoration, the process restarts its operation in the *IC* state and the first set of parameters after the set-up will correspond to a relaxed sampling $\{n_1, h_1, w_x, k_{x,1}\}$.

The upper part of the control interval of the VP_2 control chart is illustrated in Figure 5-1.

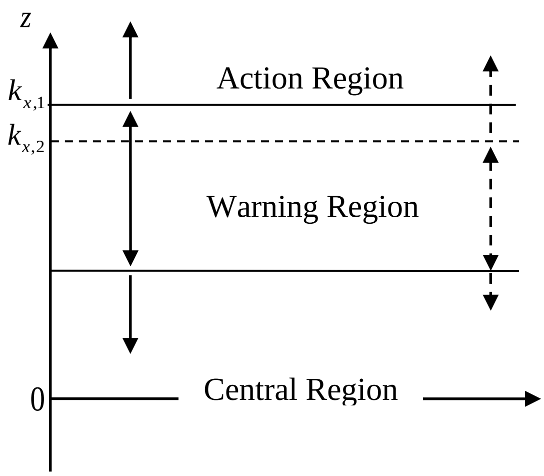


Figure 5-1: Regions of the VP_2 control chart

Regarding the actual state of the process, $Y_t = 0$ corresponds to the *IC* condition, whereas $Y_t = i$, (*i*=1,2,…,*m*), refers to an *OOC* condition due to the occurrence of the *i*-th assignable cause.

Based on the above definitions, the Markov chain has $(m+1)\times 3$ possible states (Y_t, a_t) for each possible combination of $Y_t = 0,1,...,m-1,m$ and $a_t = 0,1,2$.

Thus, the $3(m+1)\times 3(m+1)$ transition probability matrix P takes the following form:

$$
\begin{array}{ccc}
 & & \begin{array}{cccccccccccccc} k & 0 & & & 1 & & & \dots & & & m & & \\ a_t & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \end{array} \\
 & \begin{array}{cc} i & a_{t-1} \end{array} & \\
\mathrm{P} = & \begin{array}{cc} & 0 \\ 0 & 1 \\ & 2 \\ & 0 \\ 1 & 1 \\ & 2 \\ & 0 \\ \dots & 1 \\ & 2 \\ & 0 \\ m & 1 \\ & 2 \end{array} & \left[\begin{array}{c|c:c:c} & & & \\ \hline & & & \\ \hdashline & & & \\ \hdashline & & & \end{array}\right]
\end{array}
$$

Figure 5-2: Transition Probability Matrix of the VP_2 control chart

To compute all the transition probabilities $Prob_{\substack{i\,a_{t-1}\\ k\,a_t}}\left(h_q\right)$ of the *DTMC*, we first need to compute the exact expressions for all probabilities $p_{x,\substack{i\\k}}\left(h_q\right)$ of moving from any state i to any other (or the same) state k. We follow the approach described in Tagaras and Lee (1988).

In particular, if the process operates *IC* at the beginning of an interval of h_q time units, then the probability of being at state 1 at the end of that interval is denoted as $p_{x,\substack{0\\1}}\left(h_q\right)$ and equals the probability that an assignable cause occurs, times the probability that the assignable cause that occurs is the one leading to state 1, times the probability that no further assignable cause will occur thereafter:

$$p_{\underset{1}{x,0}}\left(h_q\right)=\int_0^{h_q}\frac{\lambda_{x(0\to 1)}}{v_{x,0}}\cdot v_{x,0}\cdot\exp\left(-v_{x,0}\cdot t\right)\cdot\exp\left(-v_{x,1}\cdot\left(h_q-t\right)\right)dt=$$

$$=\lambda_{x(0\to 1)}\cdot\exp\left(-v_{x,1}\cdot h_q\right)\cdot\int_0^{h_q}\exp\left(-\left(v_{x,0}-v_{x,1}\right)\cdot t\right)dt= \tag{5.1}$$

$$=\lambda_{x(0\to 1)}\cdot\exp\left(-v_{x,1}\cdot h_q\right)\cdot\frac{1-\exp\left(-\left(v_{x,0}-v_{x,1}\right)\cdot h_q\right)}{v_{x,0}-v_{x,1}}$$

The above expression is straightforwardly extended to any $i>0$, to compute the probability of moving from any state to the next, during an interval of length h_q:

$$p_{\underset{i}{x,i-1}}\left(h_q\right)=\lambda_{x(i-1\to i)}\cdot\exp\left(-v_{x,i}\cdot h_q\right)\cdot\frac{1-\exp\left(-\left(v_{x,i-1}-v_{x,i}\right)\cdot h_q\right)}{v_{x,i-1}-v_{x,i}} \tag{5.2}$$

The probability of a transition from state 0 (*IC*) to state 2 within a sampling interval h_q is equal to the sum of the probability of a direct transition from state 0 to state 2 and the probability of the process first shifting to state 1 and then to state 2 within the sampling interval h_q:

$$\begin{aligned}
p_{\underset{2}{x,0}}\left(h_q\right)&=\int_0^{h_q}\frac{\lambda_{x(0\to 2)}}{v_{x,0}}\cdot v_{x,0}\cdot\exp\left(-v_{x,0}\cdot t\right)\cdot\exp\left(-v_{x,2}\cdot\left(h_q-t\right)\right)dt+\\
&+\int_0^{h_q}\frac{\lambda_{x(0\to 1)}}{v_{x,0}}\cdot v_{x,0}\cdot\exp\left(-v_{x,0}\cdot t\right)\cdot p_{\underset{2}{1}}\left(h_q-t\right)dt\\
&=\lambda_{x(0\to 2)}\cdot\exp\left(-v_{x,2}\cdot h_q\right)\cdot\frac{1-\exp\left(-\left(v_{x,0}-v_{x,2}\right)\cdot h_q\right)}{v_{x,0}-v_{x,2}}+\\
&+\int_0^{h_q}\lambda_{x(0\to 1)}\cdot\exp\left(-v_{x,0}\cdot t\right)\cdot\lambda_{x(1\to 2)}\cdot\exp\left(-v_{x,2}\cdot\left(h_q-t\right)\right)\cdot\frac{1-\exp\left(-\left(v_{x,1}-v_{x,2}\right)\cdot\left(h_q-t\right)\right)}{v_{x,1}-v_{x,2}}dt
\end{aligned} \tag{5.3}$$

which, after some mathematical manipulation, reduces to the following expression:

$$p_{x,\underset{2}{0}}\left(h_q\right)=\lambda_{x(0\to2)}\cdot\exp\left(-v_{x,2}\cdot h_q\right)\cdot\frac{1-\exp\left(-\left(v_{x,0}-v_{x,2}\right)\cdot h_q\right)}{v_{x,0}-v_{x,2}}+$$
$$+\frac{\lambda_{x(0\to1)}\cdot\lambda_{x(1\to2)}\cdot\exp\left(-v_{x,2}\cdot h_q\right)\cdot\left(1-\exp\left(-\left(v_{x,0}-v_{x,2}\right)\cdot h_q\right)\right)}{(v_{x,1}-v_{x,2})\cdot(v_{x,0}-v_{x,2})}- \tag{5.4}$$
$$-\frac{\lambda_{x(0\to1)}\cdot\lambda_{x(1\to2)}\cdot\exp\left(-v_{x,1}\cdot h_q\right)\cdot\left(1-\exp\left(-\left(v_{x,0}-v_{x,1}\right)\cdot h_q\right)\right)}{(v_{x,1}-v_{x,2})\cdot(v_{x,0}-v_{x,1})}$$

In general, the probability of a transition from any state *i* to another state *k* $k>i+1$ can be computed recursively taking into account all the possible ways that could lead to a transition from state *i* to state *k*, by using the following equation:

$$p_{x,\underset{k}{i}}\left(h_q\right)=\int_0^{h_q}\lambda_{x(i\to k)}\cdot\exp\left(-v_{x,i}\cdot t\right)\cdot\exp\left(-v_{x,k}\cdot\left(h_q-t\right)\right)dt+$$
$$+\int_0^{h_q}\sum_{y=i+1}^{k-1}\lambda_{x(i\to j)}\cdot\exp\left(-v_{x,i}\cdot t\right)\cdot p_{x,\underset{k}{y}}\left(h_q-t\right)dt \tag{5.5}$$

In (5.5), the first integral expresses the probability of a direct shift from *i* to *k* at some time *t*, times the probability of remaining at *k* for the rest of the interval $\left(h_q-t\right)$. The second integral computes the sum of probabilities of going from state *i* to state *k* with all possible scenarios, i.e., the probability of shifting from *i* to any state *y* between *i*+1 and *k*-1 at some time *t*, times the probability of shifting from state *y* to *k* in the remainder of the interval $\left(h_q-t\right)$.

Finally, if $k=i$ $\left(\forall i\in\left[0,m\right]\right)$, the probability of no transition in h_q time units can be denoted as $p_{x,\underset{i}{i}}\left(h_q\right)$ and is equal to:

$$p_{x,\underset{i}{i}}\left(h_q\right)=\exp\left(-v_{x,i}\cdot h_q\right) \tag{5.6}$$

Obviously, when $i=m$, equation (5.6) is simplified to $p_{x,\underset{m}{m}}\left(h_q\right)=1$, (since $v_{x,m}=0$).

Thus, the exact expressions for all the transition probabilities $Prob_{\substack{i\ a_{t-1}\\ k\ a_t}}\left(h_q\right)$ of the transition probability matrix P are computed by using equations (5.1) to (5.6) and by taking into account the position of the last point plotted on the VP_2 control chart:

$$Prob_{\substack{i\ a_{t-1}\\ k\ a_t}}\left(h_q\right)=\begin{cases} p_{\substack{x,i\\ k}}\left(h_q\right)\cdot\Phi\left(w_x-\delta_k\sqrt{n_q}\right) & a_t=0\\ p_{\substack{x,i\\ k}}\left(h_q\right)\cdot\left[\Phi\left(k_{x,q}-\delta_k\sqrt{n_q}\right)-\Phi\left(w_x-\delta_k\sqrt{n_q}\right)\right] & a_t=1\\ p_{\substack{x,i\\ k}}\left(h_q\right)\cdot\left[1-\Phi\left(k_{x,q}-\delta_k\sqrt{n_q}\right)\right] & a_t=2 \end{cases} \quad (5.7)$$

Note that in the above equations, whenever $a_{t-1}=2$, i.e., when the chart issues an alarm and either a false alarm is discovered or some assignable cause is detected and eliminated, the process always restarts its operation from the *IC* state: that is, the set-up activities after a signal are always considered perfect (or they are needless in case of false alarms): as a consequence, the transition probabilities are computed thereafter by using $p_{\substack{x,0\\ k}}\left(h_q\right)$ regardless of the value of *i*:

$$Prob_{\substack{i\ 2\\ k\,a_t}}\left(h_q\right)=Prob_{\substack{0\ 0\\ k\,a_t}}\left(h_1\right) \quad (5.8)$$

The steady-state probabilities associated with the proposed Markov chain model are evaluated by solving the following linear equations system:

$$\pi_{Y_t a_t}=\sum_{Y_{t-1}=0}^{m}\sum_{a_{t-1}=0}^{2}Prob_{\substack{Y_{t-1}a_{t-1}\\ Y_t\ a_t}}\left(h_q\right)\cdot\pi_{Y_{t-1}a_{t-1}} \text{ and } \sum_{Y_t=0}^{m}\sum_{a_t=0}^{2}\pi_{Y_t a_t}=1 \quad (5.9)$$

The $\pi_{Y_t a_t}$'s represent the long-term transition probabilities for $Y_t=i$ and $a_t=v$, i.e., the process operates *IC* (if $i=0$, or under the effect of the assignable cause *i* (if $i>0$), and the plotted statistic leads to the decision $a_t=v$, ($v=0$ for relaxed parameters, $v=1$ for tightened parameters and $v=2$ for investigation).

5.3 The Economic-Statistical Design

In order to compute the average cost of a transition step *EC*, we first need to compute the expected *OOC* operation cost of the process, for state *i* being the initial state of a sampling interval of duration h_q, denoted by $K_{x,i}\left(h_q\right)$.

We will start by first defining the *OOC* cost of operating under the effect of assignable cause *m* (worst) for a whole interval of duration h_q. In this case, the *OOC* operation cost, $K_{x,m}\left(h_q\right)$, will equal the cost per time unit of operating under the effect of the cause *m*, i.e., M_m, times the duration of the interval h_q:

$$K_{x,m}\left(h_q\right) = M_m \cdot h_q \tag{5.10}$$

Given the cost $K_{x,m}\left(h_q\right)$, we can compute the quality cost of an interval that starts under the effect of the immediately "better" state, i.e., state *m*–1. If the process operates under the effect of assignable cause *m*–1 at the beginning of an interval h_q, then the *OOC* operation cost, $K_{x,m-1}\left(h_q\right)$ can be computed taking into account that the process may operate under the effect of assignable cause *m*–1 for the entire interval or a shift may occur to the only possible state left, i.e., state *m*. Thus, $K_{x,m-1}\left(h_q\right)$ will be equal to the sum of the following two terms: (a) the cost per time unit of operating under the effect of assignable cause *m*–1, M_{m-1}, multiplied by the duration of the interval h_q, times the probability of remaining under the specific effect during the interval; (b) the probability of shifting to the *m* state (only possible shift from the *m*–1 state), multiplied by the mean expected operation cost:

$$
\begin{aligned}
K_{x,m-1}\left(h_q\right) = & M_{m-1} \cdot h_q \cdot \exp\left(-v_{x,m-1} \cdot h_q\right) + \\
& + \int_0^{h_q} \frac{\lambda_{x(m-1\to m)}}{v_{x,m-1}} \cdot v_{x,m-1} \cdot \exp\left(-v_{x,m-1} \cdot t\right) \cdot \left(t \cdot M_{m-1} + K_{x,m}\left(h_q - t\right)\right) dt \\
& = M_{m-1} \cdot h_q \cdot \exp\left(-v_{x,m-1} \cdot h_q\right) + \\
& + \lambda_{x(m-1\to m)} \cdot \int_0^{h_q} \exp\left(-v_{x,m-1} \cdot t\right) \cdot \left(t \cdot M_{m-1} + \left(h_q - t\right) \cdot M_m\right) dt \\
& = M_{m-1} \cdot h_q \cdot \exp\left(-v_{x,m-1} \cdot h_q\right) + \\
& + \lambda_{x(m-1\to m)} \left[\left(M_{m-1} - M_m\right) \cdot \int_0^{h_q} t \cdot \exp\left(-v_{x,m-1} \cdot t\right) dt + M_m \cdot h_q \cdot \int_0^{h_q} \exp\left(-v_{x,m-1} \cdot t\right) dt \right]
\end{aligned}
$$

and thus,

$$
\begin{aligned}
K_{x,m-1}\left(h_q\right) = & M_{m-1} \cdot h_q \cdot \exp\left(-v_{x,m-1} \cdot h_q\right) + \\
& + \lambda_{x(m-1\to m)} \cdot \left[\begin{array}{l} \left(M_{m-1} - M_m\right) \cdot \dfrac{\left(1 - \exp\left(-v_{x,m-1} \cdot h_q\right) - v_{x,m-1} \cdot h_q \cdot \exp\left(-v_{x,m-1} \cdot h_q\right)\right)}{v_{x,m-1}^2} + \\ + M_m \cdot h_q \cdot \dfrac{1 - \exp\left(-v_{x,m-1} \cdot h_q\right)}{v_{x,m-1}} \end{array} \right]
\end{aligned} \tag{5.11}
$$

Using a similar reasoning, and with the use of expressions (5.10) and (5.11), we can compute the expected cost of an interval h_q, that starts with the process operating under the effect of state *m*–2, $K_{x,m-2}\left(h_q\right)$. In this case, $K_{x,m-2}\left(h_q\right)$ can be computed taking into account that the process may operate under the effect of assignable cause *m*–2 for the entire interval, a shift to state *m*–1 may occur, or a direct shift to state *m* may occur. Thus, $K_{x,m-2}\left(h_q\right)$ will be equal to the sum of the following three terms: (a) the cost per time unit of operating under the effect of the cause *m*–2, M_{m-2}, multiplied by the duration of the interval h_q, times the probability of remaining under the specific effect during the whole interval; (b) the probability of shifting to the *m*–1 state, multiplied by the expected operation cost which in terms consists of operating under the effect of state *m*–2 for some part of the interval and the cost $K_{x,m-1}$ for the rest of the interval; (c) the probability of shifting directly to the *m* state, multiplied by the expected operation cost of operating under the effect of state *m*–2 for some part of the interval and the cost $K_{x,m}$ for the rest of the interval:

$$
\begin{aligned}
K_{x,m-2}\left(h_q\right) &= M_{m-2}\cdot h_q\cdot\exp\left(-v_{x,m-2}\cdot h_q\right)+ \\
&+\int_0^{h_q}\lambda_{x(m-2\to m-1)}\cdot\exp\left(-v_{x,m-2}\cdot t\right)\cdot\left(t\cdot M_{m-2}+K_{x,m-1}\left(h_q-t\right)\right)dt+ \\
&+\int_0^{h_s}\lambda_{x(m-2\to m)}\cdot\exp\left(-v_{x,m-2}\cdot t\right)\cdot\left(t\cdot M_{m-2}+K_{x,m}\left(h_q-t\right)\right)dt
\end{aligned}
\tag{5.12}
$$

The three equations above (5.10), (5.11) and (5.12), are used to compute the *OOC* operation cost of the process when an interval of h_q time units starts with the operation being in the *m*, *m*–1 and *m*–2 states, respectively. In general, the *OOC* operation cost $K_{x,i}\left(h_q\right)$ for any $i<m-1$ is computed recursively from the following expression:

$$
\begin{aligned}
K_{x,i}\left(h_q\right) &= M_i\cdot h_q\cdot\exp\left(-v_{x,i}\cdot h_q\right)+ \\
&+\int_0^{h_q}\sum_{y=i+1}^{m}\left[\frac{\lambda_{x(i\to y)}}{v_{x,i}}\cdot v_{x,i}\cdot\exp\left(-v_{x,i}\cdot t\right)\cdot\left(t\cdot M_i+K_{x,y}\left(h_q-t\right)\right)\right]dt \\
&= M_i\cdot h_q\cdot\exp\left(-v_{x,i}\cdot h_q\right)+ \\
&+\int_0^{h_q}\sum_{y=i+1}^{m}\left[\lambda_{x(i\to y)}\cdot\exp\left(-v_{x,i}\cdot t\right)\cdot\left(t\cdot M_i+K_{x,y}\left(h_q-t\right)\right)\right]dt
\end{aligned}
\tag{5.13}
$$

Subsequently, the average cost of a transition step, *EC*, is the weighted average of the expected costs from all transition states and is given by:

$$
\begin{aligned}
EC = b &+ \sum_{k=0}^{m}\pi_{k\,0}\cdot\left[c\cdot n_1+K_{x,k}\left(h_1\right)\right]+\sum_{k=0}^{m}\pi_{k\,1}\cdot\left[c\cdot n_2+K_{x,k}\left(h_2\right)\right]+ \\
&+\sum_{k=0}^{m}\pi_{k\,2}\cdot\left[c\cdot n_1+K_{x,0}\left(h_1\right)+L_k\right]
\end{aligned}
\tag{5.14}
$$

Similarly, the average duration of a transition step, *ET*, is the weighted average of the durations associated with each steady-state of the chain and is given by:

$$
ET = h_1\cdot\sum_{k=0}^{m}\pi_{k\,0}+h_2\cdot\sum_{k=0}^{m}\pi_{k\,1}+\sum_{k=0}^{m}\pi_{k\,2}\cdot\left(h_1+T_k\right) \tag{5.15}
$$

5.4 Optimization Problem

The optimization problem is formulated as follows:

$$\min_{DP_q} ECT$$

$$\text{s.t.} \quad h_1, h_2, n_1, n_2, w_x, k_{x,1}, k_{x,2} > 0$$

$$h_2 \le h_1 \tag{5.16}$$

$$n_2 \ge n_1$$

$$w_x \le k_{x,2} \le k_{x,1}$$

$$n_1, n_2 \in \mathbb{N}^+$$

The minimization of *ECT* is achieved by means of a computer program developed in Fortran Power Station 4.0, which estimates the minimum *ECT* and defines the optimum design parameters of the control chart for each case.

5.5 Numerical Analysis

In this section a numerical investigation is performed to explore the potential cost savings deriving from monitoring a process with a fully adaptive control chart vs. simpler and partially adaptive (or no adaptive) charts. The numerical investigation entails 64 cases for processes where three assignable causes occur $(m = 3)$ with different process λ, δ and economic parameters (b, M_i, L_0), as presented in Table 5-1. It should be mentioned that for all cases: $\lambda_{x(i \to i+1)} = \lambda$, $\lambda_{(i \to i+2)} = \lambda / 2$, $\lambda_{(i \to i+3)} = \lambda / 4$. To limit the number of investigated scenarios, the variable sampling cost c was kept constant $(c = 1)$. The removal of each assignable cause has a different economic impact to the process: $L_1 = 200$, $L_2 = 250$ and $L_3 = 300$, while we have assumed negligible times to search and eliminate any assignable cause, i.e., $T_i = 0 \quad \forall i = 0,...,m$. Moreover, the *OOC* operation cost per time unit for each assignable cause is computed as a function of *M*, i.e., $M_1 = M$, $M_2 = 1.5 \cdot M$ and $M_3 = 2 \cdot M$.

Table 5-1: Parameter sets of the 64 numerical examples for the VP_2 control chart

Case	b	M	L_0	λ	δ_1	δ_2	δ_3	Case	b	M	L_0	λ	δ_1	δ_2	δ_3
1	0	100	100	0.01	0.125	0.25	0.375	33	0	100	100	0.01	0.5	0.75	1.0
2	0	100	200	0.01	0.125	0.25	0.375	34	0	100	200	0.01	0.5	0.75	1.0
3	0	1000	100	0.01	0.125	0.25	0.375	35	0	1000	100	0.01	0.5	0.75	1.0
4	0	1000	200	0.01	0.125	0.25	0.375	36	0	1000	200	0.01	0.5	0.75	1.0
5	5	100	100	0.01	0.125	0.25	0.375	37	5	100	100	0.01	0.5	0.75	1.0
6	5	100	200	0.01	0.125	0.25	0.375	38	5	100	200	0.01	0.5	0.75	1.0
7	5	1000	100	0.01	0.125	0.25	0.375	39	5	1000	100	0.01	0.5	0.75	1.0
8	5	1000	200	0.01	0.125	0.25	0.375	40	5	1000	200	0.01	0.5	0.75	1.0
9	0	100	100	0.1	0.125	0.25	0.375	41	0	100	100	0.1	0.5	0.75	1.0
10	0	100	200	0.1	0.125	0.25	0.375	42	0	100	200	0.1	0.5	0.75	1.0
11	0	1000	100	0.1	0.125	0.25	0.375	43	0	1000	100	0.1	0.5	0.75	1.0
12	0	1000	200	0.1	0.125	0.25	0.375	44	0	1000	200	0.1	0.5	0.75	1.0
13	5	100	100	0.1	0.125	0.25	0.375	45	5	100	100	0.1	0.5	0.75	1.0
14	5	100	200	0.1	0.125	0.25	0.375	46	5	100	200	0.1	0.5	0.75	1.0
15	5	1000	100	0.1	0.125	0.25	0.375	47	5	1000	100	0.1	0.5	0.75	1.0
16	5	1000	200	0.1	0.125	0.25	0.375	48	5	1000	200	0.1	0.5	0.75	1.0
17	0	100	100	0.01	0.25	0.5	0.75	49	0	100	100	0.01	0.5	1.5	2.5
18	0	100	200	0.01	0.25	0.5	0.75	50	0	100	200	0.01	0.5	1.5	2.5
19	0	1000	100	0.01	0.25	0.5	0.75	51	0	1000	100	0.01	0.5	1.5	2.5
20	0	1000	200	0.01	0.25	0.5	0.75	52	0	1000	200	0.01	0.5	1.5	2.5
21	5	100	100	0.01	0.25	0.5	0.75	53	5	100	100	0.01	0.5	1.5	2.5
22	5	100	200	0.01	0.25	0.5	0.75	54	5	100	200	0.01	0.5	1.5	2.5
23	5	1000	100	0.01	0.25	0.5	0.75	55	5	1000	100	0.01	0.5	1.5	2.5
24	5	1000	200	0.01	0.25	0.5	0.75	56	5	1000	200	0.01	0.5	1.5	2.5
25	0	100	100	0.1	0.25	0.5	0.75	57	0	100	100	0.1	0.5	1.5	2.5
26	0	100	200	0.1	0.25	0.5	0.75	58	0	100	200	0.1	0.5	1.5	2.5
27	0	1000	100	0.1	0.25	0.5	0.75	59	0	1000	100	0.1	0.5	1.5	2.5
28	0	1000	200	0.1	0.25	0.5	0.75	60	0	1000	200	0.1	0.5	1.5	2.5
29	5	100	100	0.1	0.25	0.5	0.75	61	5	100	100	0.1	0.5	1.5	2.5
30	5	100	200	0.1	0.25	0.5	0.75	62	5	100	200	0.1	0.5	1.5	2.5
31	5	1000	100	0.1	0.25	0.5	0.75	63	5	1000	100	0.1	0.5	1.5	2.5
32	5	1000	200	0.1	0.25	0.5	0.75	64	5	1000	200	0.1	0.5	1.5	2.5

Note that, similarly to VP_1 control scheme, the proposed VP_2 control chart is flexible enough to be associated with immediate sampling in case the statistic is found in the warning zone, i.e., if we allow $h_2 = 0.0$, then we assume that in case the tightened parameters are to be used, the next sample n_2 could be taken immediately after the previous one without allowing any time to pass.

However, in the numerical investigation, we have set a minimum value for the sampling interval (equal to 0.1) to get more realistic results since the production processes rarely cease during sampling, while the sampling procedure itself cannot realistically be instantaneous.

The economic design parameters of the VP_2 control chart, the corresponding expected quality control cost and the measures of statistical performance are presented in Tables 5-2 and 5-3 for each investigated scenario.

Table 5-2: Economic design for numerical examples 1-32: optimal control policy, cost and related statistical measures for the VP_2 control chart

Optimum Design Parameters									*Statistical Measures*						
Case	h_1	h_2	n_1	n_2	w_x	$k_{x,1}$	$k_{x,2}$	ECT_{VP2}	α	$1-\beta$	*ANOF*	ARL_0	*WARL*	*ATC*	*EATR*
1	4.5	0.4	10	11	0.1	0.7	0.5	23.31	0.2555	0.476	0.061	3.91	2.10	62.54	5.40
2	2.3	0.4	2	14	0.1	3.5	1.1	27.64	0.0548	0.150	0.031	18.25	6.65	74.85	8.18
3	1.4	0.1	11	12	0.1	0.7	0.5	69.32	0.2555	0.479	0.216	3.91	2.09	58.80	1.65
4	0.7	0.4	2	20	0.1	3.6	0.9	84.33	0.0713	0.203	0.117	14.03	4.93	69.09	2.43
5	4.9	0.4	8	9	0.1	0.5	0.3	24.43	0.3189	0.499	0.066	3.14	2.00	72.74	6.08
6	5.5	0.4	13	20	0.1	1.2	0.8	28.93	0.1454	0.343	0.032	6.88	2.92	74.65	7.98
7	1.4	0.7	9	13	0.1	0.6	0.1	74.39	0.3031	0.498	0.226	3.30	2.01	68.53	1.87
8	1.5	0.5	13	27	0.1	1.3	0.7	90.52	0.1426	0.357	0.115	7.01	2.80	69.08	2.41
9	1.8	0.4	8	9	0.1	0.6	0.3	84.31	0.2912	0.524	0.115	3.43	1.91	7.84	2.12
10	2.2	0.4	13	15	0.1	1.1	0.7	91.97	0.1649	0.433	0.054	6.06	2.31	8.43	2.72
11	0.5	0.1	12	12	0.1	0.5	0.4	233.20	0.3137	0.563	0.621	3.19	1.78	6.25	0.54
12	0.2	0.2	1	18	0.1	4.5	0.8	276.73	0.0785	0.218	0.344	12.74	4.59	7.49	0.82
13	1.8	1.0	6	8	0.1	0.6	0.1	85.56	0.2999	0.478	0.112	3.33	2.09	9.28	2.61
14	1.3	1.3	3	17	0.1	2.2	0.4	93.51	0.1094	0.297	0.049	9.14	3.37	10.42	3.76
15	0.5	0.2	9	12	0.1	0.5	0.1	243.26	0.3280	0.526	0.627	3.05	1.90	7.29	0.62
16	0.5	0.1	12	24	0.1	1.3	0.6	287.49	0.1502	0.366	0.315	6.67	2.74	7.49	0.82
17	1.9	0.1	2	9	0.2	3.1	1.3	18.44	0.0376	0.180	0.029	26.58	5.57	71.35	4.68
18	2.0	0.1	3	18	0.5	3.8	1.5	20.78	0.0193	0.184	0.012	51.70	5.45	72.03	5.36
19	0.5	0.1	2	11	0.3	3.1	1.3	52.09	0.0343	0.185	0.093	68.11	5.41	68.11	1.45
20	0.6	0.1	3	21	0.5	4.0	1.5	58.27	0.0193	0.198	0.041	51.77	5.06	68.29	1.63
21	4.2	1.6	9	14	0.1	1.3	0.9	20.45	0.1253	0.461	0.034	7.98	2.17	71.53	4.87
22	3.8	0.1	9	27	0.5	2.4	1.4	22.18	0.0285	0.291	0.009	35.10	3.43	72.19	5.53
23	1.2	0.2	9	17	0.2	1.5	1.0	60.40	0.0933	0.401	0.099	10.72	2.49	68.14	1.48
24	1.2	0.1	10	32	0.5	2.5	1.4	65.78	0.0272	0.309	0.030	36.83	3.24	68.35	1.69
25	0.6	0.3	1	7	0.1	3.7	1.2	76.97	0.0457	0.201	0.071	21.90	4.97	8.60	1.93
26	0.6	0.3	1	11	0.4	4.0	1.4	80.91	0.0247	0.174	0.034	40.46	5.75	9.08	2.41
27	0.2	0.1	2	11	0.1	2.8	1.1	192.18	0.0556	0.256	0.319	17.98	3.91	7.15	0.49
28	0.2	0.1	3	20	0.4	3.5	1.4	213.76	0.0255	0.234	0.138	39.15	4.27	7.22	0.56
29	1.6	0.8	7	11	0.1	1.3	0.8	77.41	0.1304	0.457	0.065	7.67	2.19	8.74	2.07
30	1.7	0.8	8	16	0.2	1.8	1.1	81.43	0.0672	0.389	0.031	14.88	2.57	9.18	2.52
31	0.4	0.2	9	15	0.1	1.3	0.9	204.60	0.1252	0.470	0.339	7.99	2.13	7.15	0.49
32	0.4	0.1	10	26	0.4	2.1	1.3	224.13	0.0414	0.345	0.120	24.13	2.90	7.24	0.57

Table 5-3: Economic design for numerical examples 33-64: optimal control policy, cost and related statistical measures for the VP_2 control chart

Optimum Design Parameters									*Statistical Measures*						
Case	h_1	h_2	n_1	n_2	w_x	$k_{x,1}$	$k_{x,2}$	ECT_{VP2}	α	$1\text{-}\beta$	*ANOF*	ARL_0	*WARL*	*ATC*	*EATR*
33	1.4	0.1	2	9	0.6	3.7	1.8	13.44	0.0096	0.223	0.009	104.44	4.49	69.40	2.73
34	1.5	0.1	2	13	0.7	4.8	2.1	13.91	0.0042	0.225	0.003	235.79	4.44	69.55	2.89
35	0.4	0.1	2	10	0.6	3.7	1.8	34.93	0.0096	0.241	0.029	104.44	4.15	67.52	0.85
36	0.4	0.1	2	15	0.8	4.6	2.1	36.55	0.0037	0.236	0.011	269.22	4.25	67.59	0.92
37	3.3	0.1	7	15	0.7	2.4	1.7	15.75	0.0164	0.402	0.006	61.02	2.49	69.75	3.08
38	3.2	0.1	7	19	0.8	3.0	2.0	16.09	0.0058	0.356	0.002	173.79	2.81	69.84	3.17
39	1.0	0.1	7	16	0.7	2.4	1.7	43.92	0.0164	0.410	0.020	61.02	2.44	67.64	0.98
40	1.0	0.1	7	21	0.8	2.9	2.0	45.06	0.0062	0.375	0.007	162.46	2.66	67.70	1.03
41	0.4	0.1	1	7	0.6	4.2	1.7	65.03	0.0115	0.195	0.030	86.63	5.13	7.83	1.17
42	0.4	0.1	1	10	0.8	5.0	1.9	67.19	0.0058	0.187	0.014	171.57	5.35	7.97	1.30
43	0.1	0.1	1	10	0.6	4.2	1.6	135.63	0.0141	0.265	0.133	71.14	3.77	6.99	0.33
44	0.1	0.1	1	15	0.8	4.8	1.9	143.33	0.0058	0.248	0.055	171.56	4.04	7.02	0.35
45	1.3	0.1	7	12	0.8	1.9	1.4	66.47	0.0038	0.476	0.026	26.56	2.10	7.96	1.29
46	1.3	0.1	7	15	0.8	2.4	1.7	68.15	0.0152	0.406	0.011	65.61	2.46	8.00	1.34
47	0.3	0.1	7	15	0.7	2.0	1.5	161.39	0.0032	0.480	0.115	31.38	2.09	6.99	0.32
48	0.4	0.1	8	21	0.8	2.3	1.8	167.83	0.0156	0.496	0.042	64.11	2.01	7.04	0.37
49	1.1	0.1	1	9	0.7	4.3	1.8	13.15	0.0084	0.208	0.009	119.24	4.82	69.39	2.72
50	1.5	0.1	2	14	0.8	3.9	2.1	13.42	0.0038	0.241	0.003	266.65	4.14	69.47	2.80
51	0.4	0.1	2	12	0.8	3.2	1.8	32.98	0.0079	0.258	0.023	127.14	3.87	67.53	0.86
52	0.4	0.1	2	16	0.9	3.7	2.1	34.51	0.0033	0.247	0.009	301.89	4.04	67.57	0.90
53	2.9	0.1	5	16	0.8	2.6	1.7	15.23	0.0126	0.396	0.005	79.36	2.53	69.81	3.14
54	3.0	0.1	5	19	0.8	3.1	2.0	15.56	0.0055	0.365	0.002	183.26	2.74	69.87	3.20
55	0.9	0.1	5	18	0.8	2.6	1.7	42.28	0.0126	0.409	0.017	79.36	2.45	67.67	1.00
56	0.9	0.1	6	22	0.9	3.0	2.0	43.30	0.0052	0.395	0.007	193.33	2.53	67.66	0.99
57	0.4	0.1	1	6	0.7	3.5	1.6	62.67	0.0126	0.312	0.031	79.59	4.93	7.85	1.18
58	0.4	0.1	1	10	0.9	4.1	1.9	63.98	0.0051	0.195	0.012	196.84	5.13	7.95	1.28
59	0.1	0.1	1	10	0.7	3.3	1.6	129.56	0.0127	0.266	0.121	78.44	3.75	6.99	0.33
60	0.2	0.1	4	19	0.9	2.8	1.9	137.07	0.0071	0.366	0.037	140.54	2.73	7.00	0.34
61	1.2	0.4	5	11	0.7	2.0	1.3	64.42	0.0370	0.494	0.027	27.00	2.03	8.01	1.34
62	1.2	0.1	5	16	0.9	2.7	1.7	65.87	0.0010	0.404	0.008	95.29	2.48	8.06	1.39
63	0.3	0.1	5	16	0.7	2.2	1.5	154.55	0.0252	0.471	0.091	39.68	2.12	7.00	0.34
64	0.3	0.1	6	20	0.8	2.7	1.8	159.94	0.0099	0.443	0.036	100.76	2.26	7.02	0.35

It becomes immediately evident that performing an economic optimization without any statistical constraint leads to unacceptable False Alarm probabilities for the majority of the investigated cases. In particular, the probability of *Type I* error is $\alpha > 0.02$ in 32 out of 64 cases. This results in a loss of confidence of the quality practitioner to the signals issued by the chart, while it leads to unnecessary over-adjustments of the production process.

To avoid unacceptable statistical performance for these 32 scenarios, an economic-statistical design optimization has been performed by constraining the probability of *Type I* error: $\alpha \leq 0.02$. The results of the economic-statistical optimization of the 32 cases are presented in Table 5-4.

Table 5-4: Economic-Statistical design for numerical examples with high *Type I* error: optimal control policy, cost and related statistical measures for the VP_2 control chart

Optimum Design Parameters									*Statistical Measures*						
Case	h_1	h_2	n_1	n_2	w_x	$k_{x,1}$	$k_{x,2}$	ECT_{VP2}	α	$1\text{-}\beta$	*ANOF*	ARL_0	*WARL*	*ATC*	*EATR*
1	0.5	0.1	1	4	0.5	3.1	1.5	28.67	0.0200	0.044	0.046	50.15	22.99	74.72	8.06
2	1.0	0.1	1	9	0.3	3.5	1.6	31.12	0.0200	0.060	0.026	50.09	16.75	76.28	9.61
3	0.1	0.1	1	3	0.3	3.5	1.6	80.44	0.0200	0.041	0.193	50.09	24.32	69.05	2.38
4	0.3	0.1	1	14	0.3	3.5	1.6	92.01	0.0200	0.071	0.084	50.09	14.05	69.47	2.80
5	2.3	0.1	3	28	0.3	3.5	1.6	31.77	0.0200	0.108	0.011	50.09	9.22	77.07	10.40
6	2.3	0.1	3	30	0.3	3.5	1.6	32.80	0.0200	0.115	0.010	50.09	8.72	77.48	10.82
7	0.6	0.1	3	30	0.3	3.5	1.6	100.07	0.0200	0.109	0.045	50.09	9.17	69.68	3.01
8	0.7	0.2	4	30	0.3	3.5	1.6	104.38	0.0200	0.111	0.039	50.09	9.02	70.00	3.34
9	0.2	0.1	1	3	0.5	3.0	1.5	92.11	0.0200	0.044	0.074	50.10	22.70	10.31	3.64
10	0.3	0.3	1	10	0.6	3.3	1.4	96.89	0.0200	0.070	0.040	50.01	14.27	10.80	4.13
11	0.1	0.1	1	19	0.3	3.2	1.6	279.41	0.0200	0.090	0.171	50.03	11.09	7.73	1.06
12	0.1	0.1	1	22	0.5	3.0	1.5	295.89	0.0200	0.090	0.171	50.10	11.08	7.72	1.06
13	0.7	0.6	1	24	0.4	3.5	1.5	96.25	0.0200	0.123	0.016	50.04	11.67	11.67	5.00
14	0.8	0.8	1	29	0.4	3.1	1.5	97.64	0.0199	0.140	0.013	50.19	7.14	11.99	5.32
15	0.2	0.1	2	30	0.3	3.2	1.6	323.56	0.0200	0.117	0.102	50.03	8.56	7.87	1.20
16	0.2	0.1	2	30	0.3	3.2	1.6	333.73	0.0200	0.117	0.102	50.03	8.56	7.87	1.20
17	1.2	0.1	1	9	0.3	3.9	1.6	18.81	0.0199	0.123	0.023	50.37	8.15	71.50	4.84
19	0.4	0.1	1	12	0.3	3.7	1.6	53.46	0.0199	0.144	0.067	50.28	6.92	68.22	1.55
21	3.4	0.1	6	22	0.3	3.5	1.6	21.63	0.0200	0.230	0.008	50.09	4.35	72.10	5.43
22	3.6	0.1	8	29	0.5	3.1	1.5	22.28	0.0199	0.264	0.007	50.15	3.78	72.26	5.59
23	1.0	0.1	6	25	0.3	3.5	1.6	63.66	0.0200	0.244	0.029	50.09	4.10	68.30	1.64
24	1.1	0.1	9	30	0.5	3.1	1.5	66.03	0.0199	0.270	0.024	50.15	3.70	68.42	1.75
25	0.4	0.1	1	6	0.3	3.2	1.6	77.81	0.0200	0.111	0.051	50.03	9.04	8.76	2.09
26	0.6	0.1	1	11	0.5	4.8	1.5	81.19	0.0191	0.144	0.029	52.49	6.96	9.14	2.47
27	0.1	0.1	1	15	0.5	3.3	1.5	195.68	0.0194	0.166	0.178	51.61	6.03	7.22	0.55
28	0.2	0.1	3	24	0.5	3.3	1.5	215.03	0.0194	0.227	0.102	51.61	4.41	7.26	0.59
29	0.9	0.6	2	19	0.4	3.5	1.5	78.87	0.0200	0.235	0.016	50.04	4.26	9.47	2.80
30	1.1	0.6	3	21	0.4	3.5	1.5	82.38	0.0200	0.253	0.014	50.04	3.95	9.55	2.88
31	0.3	0.1	5	26	0.3	3.2	1.6	220.45	0.0200	0.261	0.079	50.03	3.83	7.26	0.60
32	0.3	0.1	6	30	0.5	3.0	1.5	227.43	0.0200	0.271	0.074	50.10	3.69	7.28	0.62
61	1.1	0.1	4	13	0.8	2.4	1.5	64.57	0.0193	0.404	0.016	51.78	2.47	7.98	1.32
63	0.3	0.1	6	18	0.8	2.3	1.6	154.79	0.0191	0.476	0.068	52.37	2.10	7.00	0.33

The value of the optimum h_2 is equal to its minimum allowable value $(h_2 = 0.1)$ in the majority of the examined cases (90.63% and 67.19% of the scenarios with and without the *Type I* error constraint, respectively). This means that in most cases whenever an alarm is issued a sample should be collected as fast as possible after the previous one.

From the examination of the optimum design parameters, both in cases with and without the statistical constraint, a general conclusion can be drawn: for larger values of λ and/or larger values of out-of-control-operation costs, namely M, the *ECT* increases and the value of h_1 gets smaller. Moreover, it can be concluded that δ and *ECT* are inversely related variables. The logical explanation of the two aforementioned conclusions is that for large M costs, samples must be collected more frequently to limit the out-of-control operation period of the process; while, for large δ effects, assignable causes can be more easily identified by the control chart and, as a result, *ECT* gets smaller. Finally, the higher the occurrence rates of the assignable causes (λ), the smaller the sampling intervals and the lower the upper warning limit and control limits, in order for the control chart to detect quickly the more frequent deterioration of the process performance.

Additionally, the *Type I* error probability α and the *ANOF* are greater in cases with higher occurrence rates and lower effects of the assignable causes on the process mean, because of the high frequency of the assignable cause occurrence and the difficulty of the control chart to distinguish between true and false alarms. Moreover, it is apparent from the definitions of *ATC* and *EATR*, and verified by the numerical results, that the values of these statistical measures are inversely related to the values of λ.

In order to evaluate the cost savings associated with the implementation of the one-sided VP_2 control chart, a cost comparison is performed with the following partially adaptive charts: (a) *FP* control chart; (b) *VSS* control chart; (c) *VSI* control chart; (d) *VSSI* control chart. The obtained results are shown in Tables 5-5 and 5-6 and clearly demonstrate the economic superiority associated with the implementation of the VP_2 control chart.

Table 5-5: Economic comparison between the VP_2 control chart and other less adaptive control charts. Numerical examples 1-32

	ECT					$\frac{FP-VP_2}{FP}$	$\frac{VSS-VP_2}{VSS}$	$\frac{VSI-VP_2}{VSI}$	$\frac{VSSI-VP_2}{VSSI}$
Case	*FP*	*VSS*	*VSI*	*VSSI*	VP_2	(%)	(%)	(%)	(%)
1	30.18	30.18	29.74	29.74	28.67	5.00	5.00	3.60	3.60
2	36.33	35.61	35.13	35.04	31.12	14.34	12.61	11.41	11.19
3	100.44	97.58	91.17	90.55	80.44	19.91	17.57	11.77	11.17
4	117.41	115.94	108.07	107.52	92.01	21.63	20.64	14.86	14.43
5	43.83	39.34	37.42	36.08	31.77	27.52	19.24	15.10	11.95
6	44.10	40.00	38.03	36.72	32.80	25.62	18.00	13.75	10.68
7	149.42	126.11	121.60	113.80	100.07	33.03	20.65	17.71	12.07
8	150.31	128.25	123.22	120.80	104.38	30.56	18.61	15.29	13.59
9	99.03	99.03	96.52	96.52	92.11	6.99	6.99	4.57	4.57
10	109.97	107.06	103.86	103.86	96.89	11.89	9.50	6.71	6.71
11	386.64	349.12	342.48	339.88	279.41	27.73	19.97	18.42	17.79
12	400.92	363.79	356.96	354.52	295.89	26.20	18.66	17.11	16.54
13	117.11	113.16	107.73	106.11	96.25	17.81	14.94	10.66	9.29
14	117.40	113.86	108.34	106.80	97.64	16.83	14.25	9.88	8.58
15	438.32	393.45	392.59	372.88	323.56	26.18	17.76	17.58	13.23
16	441.06	400.59	396.11	377.05	333.73	24.33	16.69	15.75	11.49
17	25.15	22.69	21.22	20.27	18.81	25.21	17.10	11.36	7.20
18	26.37	23.77	22.20	21.19	20.78	21.20	12.58	6.40	1.93
19	75.98	66.02	64.62	59.72	53.46	29.64	19.02	17.27	10.48
20	80.44	69.50	67.18	62.46	58.27	27.56	16.16	13.26	6.71
21	27.22	25.13	23.76	22.64	21.63	20.54	13.93	8.96	4.46
22	27.54	25.58	24.27	23.16	22.28	19.10	12.90	8.20	3.80
23	82.82	73.75	71.21	66.55	63.66	23.13	13.68	10.60	4.34
24	83.90	75.43	72.92	68.37	66.03	21.30	12.46	9.45	3.42
25	87.33	82.89	82.38	81.30	77.81	10.90	6.13	5.55	4.29
26	88.43	85.52	83.10	82.04	81.19	8.19	5.06	2.30	1.04
27	257.16	226.95	224.71	213.82	195.68	23.91	13.78	12.92	8.48
28	263.93	237.75	235.97	220.66	215.03	18.53	9.56	8.87	2.55
29	91.30	89.93	83.89	82.80	78.87	13.61	12.30	5.98	4.75
30	91.91	90.72	84.71	83.73	82.38	10.37	9.19	2.75	1.61
31	272.40	251.32	246.49	232.96	220.45	19.07	12.28	10.56	5.37
32	275.43	256.05	250.71	237.76	227.43	17.43	11.18	9.29	4.34

Table 5-6: Economic comparison between the VP_2 control chart and other less adaptive control charts. Numerical examples 33-64

Case	ECT *FP*	*VSS*	*VSI*	*VSSI*	VP_2	$\frac{FP-VP_2}{FP}$ (%)	$\frac{VSS-VP_2}{VSS}$ (%)	$\frac{VSI-VP_2}{VSI}$ (%)	$\frac{VSSI-VP_2}{VSSI}$ (%)
33	16.65	15.51	14.02	13.69	13.44	19.28	13.35	4.14	1.83
34	17.15	16.44	14.89	14.36	13.91	18.89	15.39	6.58	3.13
35	45.45	41.44	39.18	36.71	34.93	23.15	15.71	10.85	4.85
36	47.12	44.57	41.84	38.55	36.55	22.43	17.99	12.64	5.19
37	18.03	17.56	16.30	15.87	15.75	12.65	10.31	3.37	0.76
38	18.40	18.00	16.83	16.32	16.09	12.55	10.61	4.40	1.41
39	49.83	48.12	45.87	44.31	43.92	11.86	8.73	4.25	0.88
40	51.11	49.63	47.69	45.71	45.06	11.84	9.21	5.51	1.42
41	71.83	69.70	68.19	65.97	65.03	9.47	6.70	4.63	1.42
42	73.00	71.83	69.12	68.41	67.19	7.96	6.46	2.79	1.78
43	166.46	156.27	150.59	142.37	135.63	18.52	13.21	9.93	4.73
44	171.85	165.45	157.44	150.23	143.33	16.60	13.37	8.96	4.59
45	75.77	75.11	68.23	67.43	66.47	12.27	11.50	2.58	1.42
46	76.56	75.99	69.32	68.63	68.15	10.98	10.32	1.69	0.70
47	180.30	175.96	168.02	163.98	161.39	10.49	8.28	3.95	1.58
48	184.35	180.00	172.90	168.84	167.83	8.96	6.76	2.93	0.60
49	15.89	14.19	13.73	13.67	13.15	17.24	7.33	4.22	3.80
50	16.61	15.02	14.59	13.97	13.42	19.21	10.65	8.02	3.94
51	43.55	37.39	37.27	34.74	32.98	24.27	11.79	11.51	5.07
52	45.76	39.96	39.11	36.45	34.51	24.58	13.64	11.76	5.32
53	17.67	16.49	16.07	15.41	15.23	13.81	7.64	5.23	1.17
54	18.09	17.04	16.64	15.82	15.56	13.99	8.69	6.49	1.64
55	48.91	44.85	44.14	42.83	42.28	13.56	5.73	4.21	1.28
56	50.32	46.78	46.03	44.10	43.30	13.95	7.44	5.93	1.81
57	67.28	65.50	64.97	64.13	62.67	6.85	4.32	3.54	2.28
58	69.41	67.57	67.06	65.63	63.98	7.82	5.31	4.59	2.51
59	159.07	142.94	141.79	134.55	129.56	18.55	9.36	8.63	3.71
60	166.26	150.58	149.32	141.71	137.07	17.56	8.97	8.20	3.27
61	73.37	71.54	66.87	65.28	64.57	11.99	9.74	3.44	1.09
62	74.43	72.66	68.06	66.58	65.87	11.50	9.34	3.22	1.07
63	176.84	165.01	164.85	157.01	154.79	12.47	6.19	6.10	1.41
64	180.88	170.46	169.79	162.61	159.94	11.58	6.17	5.80	1.64

The average cost saving achieved by implementing the proposed control chart is 17.38% compared to the *FP* control chart; 11.85% compared to the *VSS* control chart; 8.41% compared to the *VSI* control chart and 5.14% compared to the *VSSI* control chart. This improvement is greater when the *OOC* operation costs, namely M , are large and λ are small.

Finally, to test the economic superiority associated with the design of the VP_2 control chart optimized in presence of multiple assignable causes, we have computed the expected quality control cost $ECTs$ associated with the implementation of the same *VP* control chart, denoted as $VP_{2,s}$, with design parameters optimized for monitoring one assignable cause with average size $\delta_{av} = \sum_{i=1}^{m} \delta_i \Big/ m \quad (m=3)$. The results are given in Table 5-7.

It is evident from Table 5-7 that the erroneous consideration of only one assignable cause with average shift size δ_{av} charges the process with a significant additional cost. Specifically, the average cost saving associated with implementing the proposed model is 18.78%. As expected, this saving is greater for scenarios where M costs are large, λ are small and the relative difference between the three assignable causes is greater, i.e., $\delta_1 = 0.5$, $\delta_2 = 1.5$, $\delta_3 = 2.5$.

Table 5-7: Economic comparison between the VP_2 control chart in case of multiple assignable causes and in case of a single assignable cause

Case	h_1	h_2	n_1	n_2	w_x	$k_{x,1}$	$k_{x,2}$	ECT $VP_{2,s}$	ECT VP_2	$\frac{VP_{2,s}-VP_2}{VP_{2,s}}$ (%)	Case	h_1	h_2	n_1	n_2	w_x	$k_{x,1}$	$k_{x,2}$	ECT $VP_{2,s}$	ECT VP_2	$\frac{VP_{2,s}-VP_2}{VP_{2,s}}$ (%)
1	5.1	0.1	11	16	0.1	1.3	1.0	24.71	23.31	5.67	33	1.8	0.1	2	8	0.6	3.8	2.1	14.55	13.44	7.63
2	5.4	0.1	11	26	0.1	2.3	1.4	29.83	27.64	7.34	34	2.2	0.1	3	10	0.7	3.9	2.3	15.29	13.91	9.03
3	1.6	0.1	12	18	0.1	1.3	1.0	76.21	69.32	9.04	35	0.7	0.1	3	9	0.6	3.1	2.0	39.23	34.93	10.96
4	1.5	0.1	11	30	0.2	2.4	1.4	98.27	84.33	14.19	36	0.7	0.1	3	11	0.7	3.6	2.3	42.36	36.55	13.72
5	7.5	0.1	19	19	0.1	0.7	0.7	25.34	24.43	3.59	37	4.6	0.1	8	13	0.8	2.5	2.1	16.93	15.75	6.97
6	7.8	0.1	23	30	0.2	1.6	1.3	31.03	28.93	6.77	38	4.4	0.1	8	15	0.9	2.9	2.3	17.50	16.09	8.06
7	2.2	0.3	19	19	0.1	0.8	0.8	77.66	74.39	4.21	39	1.4	0.1	8	13	0.8	2.4	2.0	46.32	43.92	5.18
8	2.4	0.1	25	30	0.2	1.6	1.3	99.53	90.52	9.05	40	1.4	0.1	8	16	0.9	2.8	2.3	48.89	45.06	7.83
9	2.3	0.1	12	13	0.1	0.8	0.8	85.73	84.31	1.66	41	0.9	0.1	3	8	0.5	2.8	1.9	66.62	65.03	2.39
10	2.8	0.1	17	22	0.1	1.4	1.1	94.11	91.97	2.27	42	0.8	0.1	3	10	0.7	3.4	2.1	68.40	67.19	1.77
11	0.6	0.1	15	18	0.1	1.0	0.8	251.20	233.20	7.17	43	0.3	0.1	5	11	0.6	2.3	1.8	150.61	135.63	9.95
12	0.7	0.1	19	30	0.1	1.6	1.2	300.47	276.73	7.90	44	0.3	0.1	6	15	0.9	2.6	2.1	161.31	143.33	11.15
13	2.9	0.7	15	15	0.1	0.4	0.3	88.53	85.56	3.35	45	1.7	0.1	9	12	0.9	2.1	1.8	72.40	66.47	8.19
14	3.5	0.1	23	23	0.1	1.1	1.1	96.78	93.51	3.38	46	1.8	0.1	9	14	0.9	2.4	2.1	74.40	68.15	8.40
15	0.7	0.1	17	17	0.1	0.8	0.7	254.91	243.26	4.57	47	0.5	0.1	10	13	0.9	2.0	1.8	170.63	161.39	5.42
16	0.8	0.1	26	30	0.2	1.4	1.2	305.86	287.49	6.01	48	0.5	0.1	11	17	1.0	2.4	2.1	178.17	167.83	5.80
17	2.8	0.1	4	13	0.4	3.0	1.8	21.38	18.44	13.75	49	1.2	0.1	1	3	0.8	3.9	2.5	30.09	13.15	56.30
18	3.1	0.1	5	17	0.5	3.5	2.1	24.40	20.78	14.84	50	1.2	0.1	1	4	0.9	4.5	2.8	31.61	13.42	57.55
19	0.9	0.1	5	15	0.5	2.7	1.7	66.47	52.09	21.63	51	0.4	0.1	1	4	0.8	3.6	2.5	130.45	32.98	74.72
20	0.9	0.1	5	19	0.6	3.4	2.0	80.05	58.27	27.21	52	0.4	0.1	1	4	0.9	4.0	2.6	171.03	34.51	79.82
21	5.3	0.1	12	18	0.6	2.0	1.7	23.45	20.45	12.79	53	3.6	0.1	4	6	1.3	2.9	2.6	30.66	15.23	50.33
22	5.2	0.1	12	24	0.7	2.6	2.0	26.34	22.18	15.79	54	3.6	0.1	4	6	1.3	3.1	2.7	33.23	15.56	53.17
23	1.6	0.1	12	19	0.6	2.0	1.7	71.19	60.40	15.16	55	1.1	0.1	4	6	1.3	2.7	2.5	133.67	42.28	68.37
24	1.6	0.1	12	25	0.7	2.6	2.0	84.13	65.78	21.81	56	1.1	0.1	4	7	1.4	3.0	2.7	175.87	43.30	75.38
25	1.1	0.1	4	11	0.3	2.4	1.6	77.02	76.97	0.06	57	0.4	0.1	1	4	0.9	3.6	2.4	71.53	62.67	12.39
26	1.2	0.1	5	16	0.5	2.9	1.8	81.27	80.91	0.44	58	0.4	0.1	1	4	0.9	4.0	2.6	73.84	63.98	13.35
27	0.4	0.1	8	15	0.1	1.9	1.5	215.39	192.18	10.78	59	0.2	0.1	3	6	1.2	2.6	2.3	284.12	129.56	54.40
28	0.4	0.1	9	22	0.6	2.4	1.8	248.11	213.76	13.84	60	0.2	0.1	3	6	1.2	2.8	2.5	327.91	137.07	58.20
29	2.1	0.1	12	15	0.5	1.6	1.5	81.72	77.41	5.27	61	1.3	0.1	4	6	1.3	2.6	2.4	78.13	64.42	17.55
30	2.2	0.1	15	21	0.7	2.1	1.7	82.25	81.43	1.00	62	1.3	0.1	4	6	1.3	2.8	2.5	79.79	65.87	17.45
31	0.6	0.1	15	19	0.6	1.7	1.5	227.48	204.60	10.06	63	0.4	0.1	5	6	1.4	2.4	2.3	286.77	154.55	46.11
32	0.6	0.1	17	26	0.8	2.1	1.8	254.96	224.13	12.09	64	0.4	0.1	5	7	1.4	2.7	2.5	331.15	159.94	51.70

6. DEALING WITH MULTIPLE QUALITY SHIFTS AFFECTING BOTH LOCATION AND SCALE

This chapter deals with the complicated problem of multiple assignable causes that affect both the process location and variability, and the issues that arise. Moreover, it lays the ground for better understanding of the models that will be described in the following chapters.

Firstly, an approach for the computation of the probability of a process transition from one state to another, based on the independence between quality shifts that affect the location and those which affect the process scale, is presented.

However, the aforementioned approach is difficult to employ in cases where the transition rate and/or the failure rate, for integrated quality and maintenance models, depend on both the state of the location and scale of the process and cannot be independently examined. To overcome this issue, the expressions for the computation of the probability of a process transition based on a different approach are provided. Furthermore, a recursive formula which dictates the sequence for the computation of the probability of every possible process transition is given.

Another complicated issue is the computation of the *OOC* operation cost in the presence of multiple assignable causes affecting both the process location and variability. Two different approaches are presented in this chapter, the “depth-first search” and “breadth-first search” method, and the expressions for the computation of the *OOC* operation cost for each method are provided. Finally, an evaluation of the two aforementioned methods is presented.

6.1 Probability of a process transition from one actual state to another actual state

6.1.1 The “Independence” Method

In order to compute every probability for the process moving from any actual state to any other actual state, the fact that the two sets of assignable causes affect independently the mean and the standard deviation of the process can be exploited.

The computation of the probability of a process transition in case multiple assignable causes may affect only the process mean is presented in Chapter 5 (equation (5.5)). In a similar manner, the respective expression for the computation of a process transition in case only the standard deviation is affected by multiple assignable causes can be easily derived. Consequently, based on the fact that the process mean and the standard deviation of the process are affected independently, the probability of a transition from state $i \geq 0$ to any other state $k \geq i+1$ (process mean) and from state $j \geq 0$ to another state $l \geq j+1$ (standard deviation), can be computed recursively from the following expression:

$$p_{\substack{(i,j)\\(k,l)}}\left(h_q\right) = p_{\substack{x,i\\k}}\left(h_q\right) \cdot p_{\substack{s,j\\l}}\left(h_q\right) = \left(\begin{array}{l} \int_0^{h_q} \lambda_{x(i\to k)} \cdot \exp\left(-v_{x,i} \cdot t\right) \cdot \exp\left(-v_{x,k} \cdot \left(h_q - t\right)\right) dt + \\ + \int_0^{h_q} \sum_{y=i+1}^{k-1} \lambda_{x(i\to y)} \cdot \exp\left(-v_{x,i} \cdot t\right) \cdot p_{\substack{x,y\\k}}\left(h_q - t\right) dt \end{array} \right) \cdot$$

$$\cdot \left(\begin{array}{l} \int_0^{h_q} \lambda_{s(j\to l)} \cdot \exp\left(-v_{s,j} \cdot t\right) \cdot \exp\left(-v_{s,l} \cdot \left(h_q - t\right)\right) dt + \\ + \int_0^{h_q} \sum_{z=j+1}^{l-1} \lambda_{s(j\to z)} \cdot \exp\left(-v_{s,j} \cdot t\right) \cdot p_{\substack{s,z\\l}}\left(h_q - t\right) dt \end{array} \right) \tag{6.1}$$

Obviously, if $i = k$ $\left(j = l\right)$ the aforementioned probability reduces to:

$$p_{\substack{(k,j)\\(k,l)}}\left(h_q\right) = \exp\left(-v_{x,k} \cdot h_q\right) \cdot \left(\begin{array}{l} \int_0^{h_q} \lambda_{s(j\to l)} \cdot \exp\left(-v_{s,j} \cdot t\right) \cdot \exp\left(-v_{s,l} \cdot \left(h_q - t\right)\right) dt + \\ + \int_0^{h_q} \sum_{z=j+1}^{l-1} \lambda_{s(j\to z)} \cdot \exp\left(-v_{s,j} \cdot t\right) \cdot p_{\substack{s,z\\l}}\left(h_q - t\right) dt \end{array} \right) \tag{6.2}$$

$$\left(p_{\substack{(i,l)\\(k,l)}}\left(h_q\right) = \exp\left(-v_{s,l} \cdot h_q\right) \cdot \left(\begin{array}{l} \int_0^{h_q} \lambda_{x(i\to k)} \cdot \exp\left(-v_{x,i} \cdot t\right) \cdot \exp\left(-v_{x,k} \cdot \left(h_q - t\right)\right) dt + \\ + \int_0^{h_q} \sum_{y=i+1}^{k-1} \lambda_{x(i\to y)} \cdot \exp\left(-v_{x,i} \cdot t\right) \cdot p_{\substack{x,y\\k}}\left(h_q - t\right) dt \end{array} \right) \right) \tag{6.3}$$

The probability of no transition, in h_q time units can be denoted $p_{\substack{(i,j)\\(i,j)}}\left(h_q\right)$ and is equal to

$$p_{\substack{(i,j)\\(i,j)}}\left(h_q\right)=\exp\left(-\left(v_{x,i}+v_{s,j}\right)\cdot h_q\right) \tag{6.4}$$

It is obvious that when $i=m$ and $j=r$, then $p_{\substack{(m,r)\\(m,r)}}\left(h_q\right)=1$.

6.1.2 The "Step-by-Step" Method

A requisite for the process moving from one state (i,j) to another state (k,l) within a sampling interval of duration h_q is the occurrence of at least one assignable cause which would change the initial state of the process. The probability that an assignable cause occurs within the interval is equal to: $\int_0^{h_q}\left(v_{x,i}+v_{s,j}\right)\cdot\exp\left(-\left(v_{x,i}+v_{s,j}\right)\cdot t\right)dt\ =1-\exp\left(-\left(v_{x,i}+v_{s,j}\right)\cdot h_q\right)$.

Nevertheless, in order for the process to move from state (i,j) to (k,l), in case $i\neq k$ and $j\neq l$, more than one assignable causes may be the first to occur. For example, for a transition from state $(0,0)$ to $(1,1)$ the assignable cause that affects the process mean may occur first and then the one that affects the standard deviation and vice versa.

Apparently, every assignable cause *y* between the initial state *i* and the final state *k* of the process mean $\left(y\in(i,k]\right)$ and every assignable cause *z* between the initial state *j* and the final state *l* of the standard deviation $\left(z\in(j,l]\right)$ may be the precedent assignable cause, which changes the state of the process from (i,j) either to (y,j) or (i,z).

Consequently, the probability for the process moving from state (i,j) to (k,l) equals the probability that an assignable cause occurs, times the probability that the first assignable cause that occurs leads either to state $(y,j)\ \ \forall y\in(i,k]$, if it affects the process mean or $(i,z)\ \ \forall z\in(j,l]$, if it affects the standard deviation, times the probability for the process moving to the final state (k,l) in the remainder of the interval $\left(h_q - t\right)$. In general, the probability $p_{\substack{(i,j)\\(k,l)}}(h_q)$ can be computed recursively from the following expression:

$$\begin{aligned} p_{\substack{(i,j)\\(k,l)}}(h_q) = & \int_0^{h_q} \sum_{y=i+1}^{k} \left(v_{x,i} + v_{s,j}\right) \cdot \exp\left(-\left(v_{x,i} + v_{s,j}\right) \cdot t\right) \cdot \frac{\lambda_{x(i \to y)}}{\left(v_{x,i} + v_{s,j}\right)} \cdot p_{\substack{(y,j)\\(k,l)}}(h_q - t)dt + \\ & + \int_0^{h_q} \sum_{z=j+1}^{l} \left(v_{x,i} + v_{s,j}\right) \cdot \exp\left(-\left(v_{x,i} + v_{s,j}\right) \cdot t\right) \cdot \frac{\lambda_{s(j \to z)}}{\left(v_{x,i} + v_{s,j}\right)} \cdot p_{\substack{(i,z)\\(k,l)}}(h_q - t)dt \end{aligned} \tag{6.5}$$

which after some mathematical manipulation can be simplified to the following equation:

$$\begin{aligned} p_{\substack{(i,j)\\(k,l)}}(h_q) = & \int_0^{h_q} \sum_{y=i+1}^{k} \lambda_{x(i \to y)} \cdot \exp\left(-\left(v_{x,i} + v_{s,j}\right) \cdot t\right) \cdot p_{\substack{(y,j)\\(k,l)}}(h_q - t)dt + \\ & + \int_0^{h_q} \sum_{z=j+1}^{l} \lambda_{s(j \to z)} \cdot \exp\left(-\left(v_{x,i} + v_{s,j}\right) \cdot t\right) \cdot p_{\substack{(i,z)\\(k,l)}}(h_q - t)dt \end{aligned} \tag{6.6}$$

Obviously, if $k = i$ $\left(l = j\right)$ the aforementioned probability reduces to:

$$p_{\substack{(k,j)\\(k,l)}}(h_q) = \int_0^{h_q} \sum_{z=j+1}^{l} \lambda_{s(j \to z)} \cdot \exp\left(-\left(v_{x,k} + v_{s,j}\right) \cdot t\right) \cdot p_{\substack{(k,z)\\(k,l)}}(h_q - t)dt \tag{6.7}$$

$$\left(p_{\substack{(i,l)\\(k,l)}}(h_q) = \int_0^{h_q} \sum_{y=i+1}^{k} \lambda_{x(i \to y)} \cdot \exp\left(-\left(v_{x,i} + v_{s,l}\right) \cdot t\right) \cdot p_{\substack{(y,l)\\(k,l)}}(h_q - t)dt \right) \tag{6.8}$$

Moreover, the probability of no transition $p_{\substack{(i,j)\\(i,j)}}\left(h_q\right)$ is the same regardless of the employed method and can be derived from equation (6.4). Also, $p_{\substack{(m,r)\\(m,r)}}\left(h_q\right) = 1$.

It should be noted that the two methods, i.e., the “independence” and the “step-by-step” method, could be equivalently utilized for the computation of the probability for a process transition from one actual state to another. Moreover, the exact same equations can be employed in multivariate processes by substituting λ_x with λ_{mv}, λ_s with λ_{cm}, v_x with v_{mv} and v_s with v_{cm} and by taking into account that in multivariate processes a transition of the process to any other state is feasible providing that no quality characteristic is improved (see Section 9.2).

However, the latter, i.e the "step-by-step" method, is preferable in cases where either the transition rate or the failure rate, as will be discussed in detail in Chapter 8, are contingent upon both the state of the process mean and the state of the standard deviation and, so, could not be independently examined.

6.1.3 Recursive Formula

Regardless of the approach utilized for the computation of the probability for a process transition and because both equations (equation (6.1) and (6.6)) are recursive, the computation of the probability for a process transition from one actual state to another actual state prerequisites the computation of several other probabilities. For example, in order to compute $p_{\substack{(0,0)\\(2,0)}}\left(h_q\right)$, it is necessary to employ $p_{\substack{(0,0)\\(1,0)}}\left(h_q-t\right)$, so $p_{\substack{(0,0)\\(1,0)}}\left(h_q\right)$ should be computed first.

In order to explain that clearly, we employ as an example the computation of the probability for a process transition from state $(0,0)$ to $(2,2)$, when two quality shifts may occur that affect the mean and two the standard deviation of the process $(m=r=2)$ and we demonstrate an easy-to-implement procedure to do so.

Firstly, in order to compute $p_{\substack{(0,0)\\(2,2)}}\left(h_q\right)$, a matrix with all the possible states between the initial state, i.e., $(0,0)$, and the final one, i.e., $(2,2)$, should be created. Then, in this upper triangular matrix, the computation of the probabilities should begin from the last row moving to the first one and the probabilities within each row should be computed from the left to the right. The matrix for our example and the desirable sequence of the computation of probabilities are presented below:

$$
\begin{array}{cccccccccc}
(0,0) & (0,1) & (0,2) & (1,0) & (1,1) & (1,2) & (2,0) & (2,1) & (2,2) & \\
p_{\substack{(0,0)\\(0,0)}} & p_{\substack{(0,0)\\(0,1)}} & p_{\substack{(0,0)\\(0,2)}} & p_{\substack{(0,0)\\(1,0)}} & p_{\substack{(0,0)\\(1,1)}} & p_{\substack{(0,0)\\(1,2)}} & p_{\substack{(0,0)\\(2,0)}} & p_{\substack{(0,0)\\(2,1)}} & p_{\substack{(0,0)\\(2,2)}} & (0,0) \\
 & p_{\substack{(0,1)\\(0,1)}} & p_{\substack{(0,1)\\(0,2)}} & - & p_{\substack{(0,1)\\(1,1)}} & p_{\substack{(0,1)\\(1,2)}} & - & p_{\substack{(0,1)\\(2,1)}} & p_{\substack{(0,1)\\(2,2)}} & (0,1) \\
 & & p_{\substack{(0,2)\\(0,2)}} & - & - & p_{\substack{(0,2)\\(1,2)}} & - & - & p_{\substack{(0,2)\\(2,2)}} & (0,2) \\
 & & & p_{\substack{(1,0)\\(1,0)}} & p_{\substack{(1,0)\\(1,1)}} & p_{\substack{(1,0)\\(1,2)}} & p_{\substack{(1,0)\\(2,0)}} & p_{\substack{(1,0)\\(2,1)}} & p_{\substack{(1,0)\\(2,2)}} & (1,0) \\
 & & & & p_{\substack{(1,1)\\(1,1)}} & p_{\substack{(1,1)\\(1,2)}} & - & p_{\substack{(1,1)\\(2,1)}} & p_{\substack{(1,1)\\(2,2)}} & (1,1) \\
 & & & & & p_{\substack{(1,2)\\(1,2)}} & - & - & p_{\substack{(1,2)\\(2,2)}} & (1,2) \\
 & & & & & & p_{\substack{(2,0)\\(2,0)}} & p_{\substack{(2,0)\\(2,1)}} & p_{\substack{(2,0)\\(2,2)}} & (2,0) \\
 & & & & & & & p_{\substack{(2,1)\\(2,1)}} & p_{\substack{(2,1)\\(2,2)}} & (2,1) \\
 & & & & & & & & p_{\substack{(2,2)\\(2,2)}} & (2,2)
\end{array}
$$

The probabilities of all the possible transitions should be computed from equations (6.1) to (6.4) (or equivalently equations (6.6) to (6.8) and (6.4)) depending on whether both the process mean and standard deviation are affected, only the mean, only the standard deviation or none of them, respectively. The equations utilized for the probability of each possible transition are presented in the following matrix:

$$
\begin{array}{cccccccccc}
(0,0) & (0,1) & (0,2) & (1,0) & (1,1) & (1,2) & (2,0) & (2,1) & (2,2) & \\
(6.4) & (6.3) & (6.3) & (6.2) & (6.1) & (6.1) & (6.2) & (6.1) & (6.1) & (0,0) \\
 & (6.4) & (6.3) & - & (6.2) & (6.1) & - & (6.2) & (6.1) & (0,1) \\
 & & (6.4) & - & - & (6.2) & - & - & (6.2) & (0,2) \\
 & & & (6.4) & (6.3) & (6.3) & (6.2) & (6.1) & (6.1) & (1,0) \\
 & & & & (6.4) & (6.3) & - & (6.2) & (6.1) & (1,1) \\
 & & & & & (6.4) & - & - & (6.2) & (1,2) \\
 & & & & & & (6.4) & (6.3) & (6.3) & (2,0) \\
 & & & & & & & (6.4) & (6.3) & (2,1) \\
 & & & & & & & & (6.4) & (2,2)
\end{array}
$$

6.2 Out-of-control Operation Cost

6.2.1 The “Depth-First Search” Method

The cost of a transition step depends on the values of a_{t-1}, Y_{t-1}, Y_t and on the exact order of the assignable causes’ occurrence within the interval, so as for the process to move from Y_{t-1} to Y_t.

In particular, in order to compute the expected *OOC* operation cost for a transition step, where $Y_{t-1}=(i,j)$ and $Y_t=(k,l)$, denoted by ECK, every possible scenario (combination of the chronological sequence of the assignable causes that occur within this interval in order for the process to move from state $Y_{t-1}=(i,j)$ to state $Y_t=(k,l)$) should be taken into account.

The *OOC* operation cost for each of the possible scenarios for a process transition is denoted by $CK(fn_1+fn_2)_{\substack{(i,j)\\(k,l)}}$. The variables fn_1 and fn_2 indicate the number of the assignable causes, in each scenario, that affect the mean and the standard deviation, respectively, in order for the process to move from state $Y_{t-1}=(i,j)$ to $Y_t=(k,l)$. The process transition from one state to another, during a transition step, may occur through more than one different ways as regards the number of the assignable causes. It is apparent that $fn_1 \le (k-i)$ and $fn_2 \le (l-j)$, (fn_1, fn_2 ≥ 0).

For example, if we assume $Y_{t-1}=(0,0)$ and $Y_t=(2,2)$, the process may be shifted directly to state Y_t by the occurrence of assignable causes $i=2$ affecting the mean and $j=2$ affecting the standard deviation of the process $(fn_1=1, fn_2=1)$, with assignable cause *i* occurring earlier than *j* and vice versa. Another possible scenario is that assignable cause $i=1$ occurs first, then $i=2$ and finally $j=2$ $(fn_1=2, fn_2=1)$ or $j=2$, then $i=1$ and then $i=2$ and so on.

Consequently, for the computation of *CK*, the following parameters should be computed:

(a) The probability of a specific possible scenario to occur when $\left(fn_1+fn_2\right)$ assignable causes occur within an interval of duration h_q time units, which is denoted by $\Pr\left(h_q,\left(fn_1+fn_2\right)\right)$;

(b) The expected times of the occurrence of each assignable cause of a specific scenario, in order to compute the time the process operates under each intermediate state (where, $\tau^{(1)}$ is the expected time of occurrence for the precedent assignable cause, $\tau^{(2)}$ for the second one,..., $\tau^{(fn_1+fn_2)}$ for the last one to occur).

The aforementioned parameters for a process transition from state (i,j) to (k,l) are computed from the following expressions, which extend equations (4.12)-(4.14) to the multiple assignable causes scenario affecting both the mean and the dispersion of the process. It should be mentioned that for the computation of their values, the occurrence rate λ should be substituted by λ_x, for the assignable cause that affects the process mean and by λ_s, for the one that affects the standard deviation.

$$\begin{aligned}\Pr\left(h_q,\left(fn_1+fn_2\right)\right)_{\substack{(i,j)\\(k,l)}} &= \int_0^{h_q} \lambda_1\cdot\exp\left(-\left(v_{x,i}+v_{s,j}\right)\cdot t_1\right)\int_{t_1}^{h_q}\lambda_2\cdot\exp\left(-\left(v_{x,(1)}+v_{s,(1)}\right)\cdot\left(t_2-t_1\right)\right)\ldots\\ &\ldots\int_{t_{[(fn_1+fn_2)-1]}}^{h_q}\begin{array}{l}\lambda_{(fn_1+fn_2)}\cdot\exp\left(-\left(v_{x,[(fn_1+fn_2)-1]}+v_{s,[(fn_1+fn_2)-1]}\right)\cdot\left(t_{(fn_1+fn_2)}-t_{[(fn_1+fn_2)-1]}\right)\right)\cdot\\ \cdot\exp\left(-\left(v_{x,k}+v_{s,l}\right)\cdot\left(h_q-t_{(fn_1+fn_2)}\right)\right)dt_{(fn_1+fn_2)}\ldots dt_2dt_1\end{array}\end{aligned} \tag{6.9}$$

$$\tau^{(g)}\left(h_q\right)=\frac{\begin{bmatrix}\int_0^{h_q} t_1\lambda_1\cdot\exp\left(-\left(v_{x,i}+v_{s,j}\right)\cdot t_1\right)\ldots\\ \ldots\int_{t_1}^{h_q} t_2\lambda_2\cdot\exp\left(-\left(v_{x,(1)}+v_{s,(1)}\right)\cdot\left(t_2-t_1\right)\right)\ldots\\ \ldots\int_{t_{[(fn_1+fn_2)-1]}}^{h_q}\begin{matrix}t_{(fn_1+fn_2)}\lambda_{(fn_1+fn_2)}\cdot\exp\left(-\left(v_{x,[(fn_1+fn_2)-1]}+v_{s,[(fn_1+fn_2)-1]}\right)\cdot\left(t_{(fn_1+fn_2)}-t_{[(fn_1+fn_2)-1]}\right)\right)\cdot\\ \cdot\exp\left(-\left(v_{x,k}+v_{s,l}\right)\cdot\left(h_q-t_{(fn_1+fn_2)}\right)\right)dt_{(fn_1+fn_2)}\ldots dt_2dt_1\end{matrix}\end{bmatrix}}{\begin{bmatrix}\int_0^{h_q}\lambda_1\cdot\exp\left(-\left(v_{x,i}+v_{s,j}\right)\cdot t_1\right)\ldots\\ \ldots\int_{t_1}^{h_q}\lambda_2\cdot\exp\left(-(v_{x,(1)}+v_{s,(1)})\cdot\left(t_2-t_1\right)\right)\ldots\\ \ldots\int_{t_{[(fn_1+fn_2)-1]}}^{h_q}\begin{matrix}\lambda_{(fn_1+fn_2)}\cdot\exp\left(-\left(v_{x,[(fn_1+fn_2)-1]}+v_{s,[(fn_1+fn_2)-1]}\right)\cdot\left(t_{(fn_1+fn_2)}-t_{[(fn_1+fn_2)-1]}\right)\right)\cdot\\ \cdot\exp\left(-(v_{x,k}+v_{s,l})\cdot\left(h_q-t_{(fn_1+fn_2)}\right)\right)dt_{(fn_1+fn_2)}\ldots dt_2dt_1\end{matrix}\end{bmatrix}} \tag{6.10}$$

where, $t_u=1$, for $u=\left\{1,2,\ldots,\left(fn_1+fn_2\right)\right\}\setminus\left\{g\right\}$.

It is apparent that by assuming that $\left(fn_1+fn_2\right)$ assignable causes occur within an interval, the process operates under the effect of $\left(fn_1+fn_2+1\right)$ different intermediate states. For example, if we suppose that at the beginning of an interval the process is *IC* and two assignable causes occur, firstly $i=1$ and after that, $j=2$, then, the three intermediate states are $(0,0)$, $(1,0)$ and $(1,2)$.

Having computed previously, the expected time of the occurrence of each assignable cause, the time length that the process operates under the effect of each of the intermediate states can also be computed. So, the *OOC* operation cost *CK* of a specific scenario can be, now, computed as:

$$CK\left(h_q,\left(fn_1+fn_2\right)\right)_{\substack{(i,j)\\(k,l)}}=\Pr\left(h_q,\left(fn_1+fn_2\right)\right)_{\substack{(i,j)\\(k,l)}}\cdot\begin{pmatrix}M_{(i,j)}\cdot\tau^{(1)}+M^{(1)}\cdot\left(\tau^{(2)}-\tau^{(1)}\right)+\ldots\\ \ldots+M_{(k,l)}\cdot\left(h_q-\tau^{(fn_1+fn_2)}\right)\end{pmatrix} \tag{6.11}$$

The expected *OOC* operation cost (*ECK*) for the process transition from a specific state to another specific state, is defined as the sum of the *OOC* operation costs (*CK*) for each possible scenario as regards each possible permutation of the assignable causes that occur within the interval and result in the transition.

The number of all possible permutations when a total number of $\left(fn_1 + fn_2\right)$ assignable causes occur within a transition step, with fn_1 assignable causes affecting the process mean and fn_2 affecting the standard deviation can be computed as $\binom{(fn_1+fn_2)}{fn_1} = \binom{(fn_1+fn_2)}{fn_2} = \frac{\left(fn_1+fn_2\right)!}{fn_1!\,fn_2!}$. Consequently, for a given number of fn_1 and fn_2 assignable causes that affect the process mean and the standard deviation, respectively, within an interval, a total number of $\frac{\left(fn_1+fn_2\right)!}{fn_1!\,fn_2!}$ permutations should be taken into account for the computation of *ECK*.

For example, by assuming that $fn_1 = fn_2 = 2$, there are six possible permutations as regards the chronological sequence of each assignable cause's occurrence $\left(\frac{\left(fn_1+fn_2\right)!}{fn_1!\,fn_2!} = \frac{4!}{2!2!} = 6\right)$. If the first assignable cause that occurs within the interval and affects the process mean is denoted by $x^{(1)}$, the second one by $x^{(2)}$ and by $s^{(1)}$, $s^{(2)}$ the respective assignable causes that affect the standard deviation, then the six possible permutations are: $\left(x^{(1)}, x^{(2)}, s^{(1)}, s^{(2)}\right), \left(x^{(1)}, s^{(1)}, x^{(2)}, s^{(2)}\right), \left(x^{(1)}, s^{(1)}, s^{(2)}, x^{(2)}\right)$, $\left(s^{(1)}, s^{(2)}, x^{(1)}, x^{(2)}\right), \left(s^{(1)}, x^{(1)}, x^{(2)}, s^{(2)}\right)$ and $\left(s^{(1)}, x^{(1)}, s^{(2)}, x^{(2)}\right)$.

It is apparent that, under the assumption that only transitions to inferior states may occur, there is only one combination for the exact order of the assignable causes' occurrence when they either affect only the mean $\left(fn_1 > 0,\, fn_2 = 0\right)$ or only the standard deviation $\left(fn_1 = 0,\, fn_2 > 0\right)$ of the process $\left(\frac{\left(fn_1+fn_2\right)!}{fn_1!\,fn_2!} = 1\right)$. In such case, the expected *OOC* operation cost of a transition step *ECK* equals the *OOC* operation cost (*ECK*=*CK*).

The following brute-force algorithm is employed to define the set of the possible scenarios for a process transition from state (i, j) (root) to another specific state (k,l), if and only if $i \neq k$ and $j \neq l$, denoted by $SC_{\substack{(i,j)\\(k,l)}}$.

The *DFS* algorithm is utilized in order to traverse a directed graph made by the set of process states (nodes) *V* and the edges between them whenever a transition is possible. The adjacency matrix *E* can be easily created from the occurrence rates matrix by substituting every occurrence rate for a process transition from state (i, j) to (k,l) greater than zero with: 0, if $i \neq k$ and $j \neq l$ or $i = k$ and $j = l$; 1, if $i = k$ or $j = l$.

An easy way to present the algorithm and its implementation to our model is through the following steps:

<u>*Step 1*</u>: Every node in *V* is considered as unmarked. Set $SC = 0$.

<u>*Step 2*</u>: Select an unvisited (unmarked) node w_1 that is adjacent to the root node $w_0 = (i, j)$ $\left(w_1 = Adj(w_0)\right)$.

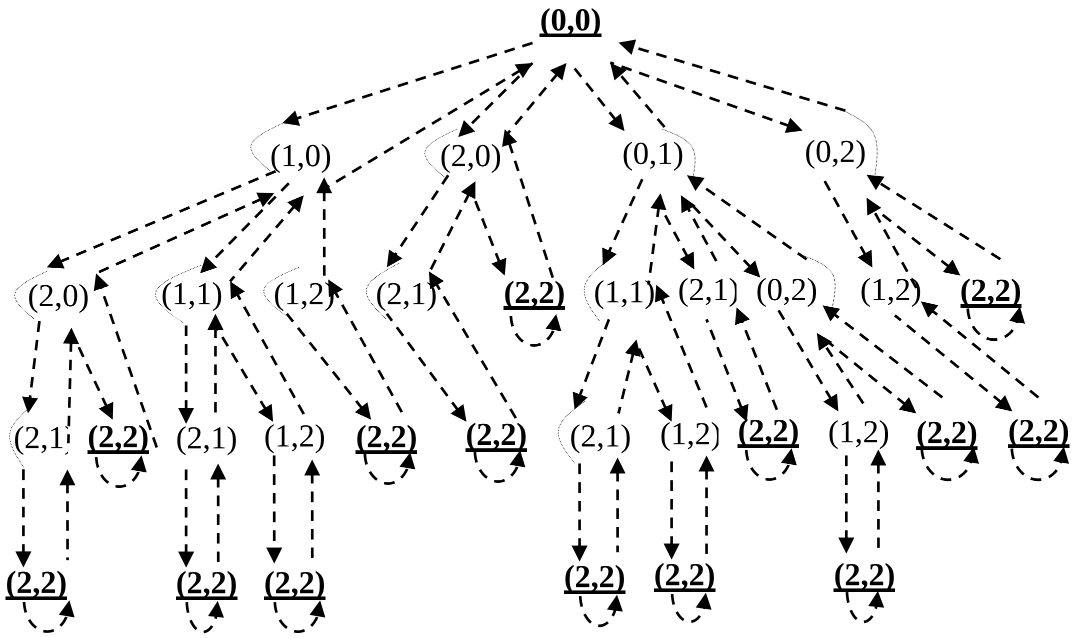
<u>*Step 3*</u>: Mark node w_1 and check whether it is the final state (k,l).

<u>*Step 4a*:</u> If yes, store this path to $SC_{\substack{(i,j)\\(k,l)}}$, go to the predecessor node w_0 and choose another unmarked and adjacent one $\left(w_1' = Adj(w_0)\right)$.

<u>*Step 4b*:</u> If not, apply the procedure (Steps 2-4) with all the adjacent to w_1, unmarked nodes $\left(w_2 = Adj(w_1)\right)$ and so on.

<u>*Step 5*</u>: Repeat until all adjacent to w_0 nodes have been visited (marked).

A visual representation of the traversal of a graph by the *DFS* algorithm is presented in Figure 6-1. Specifically, we utilize an example where state $(0,0)$ is the initial state, state $(2,2)$ the final state and $m = r = 2$. The depth ward motion of the traversal of the graph is illustrated by arrows and all the generated by the algorithm paths for our example, i.e., $SC_{\substack{(0,0)\\(2,2)}}$, are also presented.



		$SC_{\substack{(0,0)\\(2,2)}}$		
(0,0)	(1,0)	(2,0)	(2,1)	**(2,2)**
(0,0)	(1,0)	(2,0)	**(2,2)**	
(0,0)	(1,0)	(1,1)	(2,1)	**(2,2)**
(0,0)	(1,0)	(1,1)	(1,2)	**(2,2)**
(0,0)	(1,0)	(1,2)	**(2,2)**	
(0,0)	(2,0)	(2,1)	**(2,2)**	
(0,0)	(2,0)	**(2,2)**		
(0,0)	(0,1)	(1,1)	(2,1)	**(2,2)**
(0,0)	(0,1)	(1,1)	(1,2)	**(2,2)**
(0,0)	(0,1)	(2,1)	**(2,2)**	
(0,0)	(0,1)	(0,2)	(1,2)	**(2,2)**
(0,0)	(0,1)	(0,2)	**(2,2)**	
(0,0)	(0,2)	(1,2)	**(2,2)**	
(0,0)	(0,2)	**(2,2)**		

Figure 6-1: Example for *DFS* algorithm implementation

It should be mentioned that a verification of the complementation of the set of permutations for a process transition, generated by the presented algorithm, can be made through a basic principle of the adjacency matrices. Namely, the product of the multiplication of the adjacency matrix E by itself, i.e., E^2, defines the possible 2-

length paths $\left(\left(fn_1+fn_2\right)=2\right)$, E^3 the possible 3-length paths $\left(\left(fn_1+fn_2\right)=3\right)$, etc., for a process transition from one state to another specific state.

The expected *OOC* operation cost when the process is under the effect of assignable causes (i, j) at the beginning of a transition step, is denoted by $K_{(i,j)}\left(h_q\right)$ and can be, now, computed as the sum of the following three terms:

(i) The cost per time unit if no assignable cause occurs within the interval, multiplied by the duration of the interval h_q (h_1 if $a_{t-1}=0$ or 2 and h_2 if $a_{t-1}=1$), multiplied by the probability that no assignable cause occurs within the interval.

(ii) The cost per time unit in case either only the mean (the standard deviation) of the process is affected, multiplied by the expected duration the process is under the effect of each assignable cause, times the probability of each possible scenario to occur. It should be mentioned that the value of the expected *OOC* operation cost of the process in cases where the standard deviation of the process remains unaffected during the interval is denoted by $K_{x,(i,j)}\left(h_q\right)$ and its computation is based on equation (5.13). Respectively, the expected *OOC* operation cost if the process mean remains unaffected during the interval, $K_{s,(i,j)}\left(h_q\right)$, is computed from: $K_{s,(i,j)}\left(h_q\right)=M_{(i,j)}\cdot h_q\cdot\exp\left(-v_{s,j}\cdot h_q\right)+\int_0^{h_q}\sum_{z=j+1}^{r}\begin{bmatrix}\lambda_{s(j\to z)}\cdot\exp\left(-v_{s,j}\cdot t\right)\cdot\\ \left(t\cdot M_{(i,j)}+K_{s,(i,z)}\left(h_q-t\right)\right)\end{bmatrix}dt$.

(iii) The cost per time unit when both the mean and the standard deviation of the process are shifted from their initial values, multiplied by the expected duration the process is under the effect of each assignable cause, times the probability of every possible combination as regards the occurrence of the assignable causes within the interval.

For example, let us suppose that at the beginning of an interval the actual state of the process is $(i, j)=(0,1)$ and $m=r=2$. In such case, $K_{(0,1)}\left(h_q\right)$ equals the sum of the following terms:

(i) The cost per time unit if no assignable cause occurs and the process remains under the effect of assignable cause $j=1$ $\left(M_{(0,1)}\right)$ for the whole interval, multiplied by the probability that no assignable cause occurs within the interval $\left(\exp\left(-\left(v_{x,0}+v_{s,1}\right)\cdot h_q\right)\right)$, multiplied by the duration h_q.

(ii) The cost per time unit if the process mean is shifted by either assignable cause $i=1$ or $i=2$ or both, but no assignable cause that affects the standard deviation occurs, multiplied by the probability that such a scenario occurs, times the expected duration the process remains under the effect of each assignable cause. In the same manner, we should consider the cost per time unit, in case the standard deviation is further deteriorated by the occurrence of $j=2$, but the process mean remains unaffected $(i=0)$, multiplied by the probability that only $j=2$ occurs within the interval, multiplied by the expected duration that the process remains under the effect of $(i,j)=(0,1)$ and, then, under the effect of $(i,j)=(0,2)$.

(iii) The *OOC* operation cost if both the mean and the standard deviation are shifted within a transition step $\left(ECK\left(h_q,2\right)_{\substack{(0,1)\\(1,2)}}, ECK\left(h_q,2\right)_{\substack{(0,1)\\(2,2)}}, ECK\left(h_q,3\right)_{\substack{(0,1)\\(2,2)}}\right)$. $ECK\left(h_q,2\right)_{\substack{(0,1)\\(1,2)}}$ equals the sum of $CK\left(h_q,2\right)_{\substack{(0,1)\\(1,2)}}$ with $i=1$ occurring first and then $j=2$, plus $CK\left(h_q,2\right)_{\substack{(0,1)\\(1,2)}}$ with $j=2$ occurring first and then $i=1$. Similarly, $ECK\left(h_q,2\right)_{\substack{(0,1)\\(2,2)}}$ is equal to the sum of the costs if $i=2$, $j=2$ occur, plus, the cost if $j=2$ occurs first, and then $i=2$. In case three assignable causes occur within the transition step $i=1$, $i=2$, $j=2$, $ECK\left(h_q,3\right)_{\substack{(0,1)\\(2,2)}}$ equals the sum of the *OOC* operation costs, for every possible combination as regards the chronological sequence of the assignable causes' occurrence.

Subsequently, the computation of the expected *OOC* operation cost for the above example can be derived from the following expression:

$$K_{(0,1)}\left(h_q\right)=M_{(0,1)}\cdot h_q\cdot\exp\left(-\left(v_{x,0}+v_{s,1}\right)\cdot h_q\right)+$$
$$+\left(\begin{array}{l}\exp\left(-v_{s,1}\cdot h_q\right)\cdot\int_0^{h_q}\sum_{k=1}^{2}\left[\lambda_{x(0\to k)}\cdot\exp\left(-v_{x,0}\cdot t\right)\cdot\left(t\cdot M_{(0,1)}+K_{x,(k,1)}\left(h_q-t\right)\right)\right]dt+\\+\exp\left(-v_{x,0}\cdot h_q\right)\cdot\int_0^{h_q}\lambda_{s(1\to 2)}\cdot\exp\left(-v_{s,1}\cdot t\right)\cdot\left(t\cdot M_{(0,1)}+K_{s,(0,2)}\left(h_q-t\right)\right)dt\end{array}\right)+$$
$$+ECK\left(h_q,2\right)_{\substack{(0,1)\\(1,2)}}+ECK\left(h_q,2\right)_{\substack{(0,1)\\(2,2)}}+ECK\left(h_q,3\right)_{\substack{(0,1)\\(2,2)}} \tag{6.12}$$

In general, the expected *OOC* operation cost for every possible initial state when up to *m* and *r* assignable causes may occur, can be derived from the following expression:

$$K_{(i,j)}\left(h_q\right)=M_{(i,j)}\cdot h_q\cdot\exp\left(-\left(v_{x,i}+v_{s,j}\right)\cdot h_q\right)+$$
$$+\left(\begin{array}{l}\exp\left(-v_{s,j}\cdot h_q\right)\cdot\int_0^{h_q}\sum_{y=i+1}^{m}\left[\lambda_{x(i\to y)}\cdot\exp\left(-v_{x,i}\cdot t\right)\cdot\left(t\cdot M_{(i,j)}+K_{x,(y,j)}\left(h_q-t\right)\right)\right]dt+\\+\exp\left(-v_{x,i}\cdot h_q\right)\cdot\int_0^{h_q}\sum_{z=j+1}^{r}\left[\lambda_{s(j\to z)}\cdot\exp\left(-v_{s,j}\cdot t\right)\cdot\left(t\cdot M_{(i,j)}+K_{s,(i,z)}\left(h_q-t\right)\right)\right]dt\end{array}\right)+$$
$$+\sum_{y=i+1}^{m}\sum_{z=j+1}^{r}\left[\sum_{fn_1=1}^{y-i}\sum_{fn_2=1}^{z-j}ECK\left(h_q,\left(fn_1+fn_2\right)\right)_{\substack{(i,j)\\(y,z)}}\right] \tag{6.13}$$

or, equivalently

$$K_{(i,j)}\left(h_q\right)=M_{(i,j)}\cdot h_q\cdot\exp\left(-\left(v_{x,i}+v_{s,j}\right)\cdot h_q\right)+$$
$$+\left(\begin{array}{l}\exp\left(-v_{s,j}\cdot h_q\right)\cdot\int_0^{h_q}\sum_{y=i+1}^{m}\left[\lambda_{x(i\to y)}\cdot\exp\left(-v_{x,i}\cdot t\right)\cdot\left(t\cdot M_{(i,j)}+K_{x,(y,j)}\left(h_q-t\right)\right)\right]dt+\\+\exp\left(-v_{x,i}\cdot h_q\right)\cdot\int_0^{h_q}\sum_{z=j+1}^{r}\left[\lambda_{s(j\to z)}\cdot\exp\left(-v_{s,j}\cdot t\right)\cdot\left(t\cdot M_{(i,j)}+K_{s,(i,z)}\left(h_q-t\right)\right)\right]dt\end{array}\right)+$$
$$+\sum_{y=i+1}^{m}\sum_{z=j+1}^{r}\sum_{SC_{\substack{(i,j)\\(y,z)}}}\Pr\left(h_q,\left(fn_1+fn_2\right)\right)_{\substack{(i,j)\\(y,z)}}\cdot\left[\begin{array}{l}M_{(i,j)}\cdot\tau^{(1)}\left(h_q\right)+\\+M^{(1)}\cdot\left(\tau^{(2)}\left(h_q\right)-\tau^{(1)}\left(h_q\right)\right)+\ldots\\\ldots+M_{(y,z)}\cdot\left(\tau_{OOC/(i,j)}\left(h_q\right)-\tau^{(fn_1+fn_2)}\left(h_q\right)\right)\end{array}\right] \tag{6.14}$$

6.2.2 The “Breadth-First Search” Method

This method is based on the same logic compared to the “step-by-step” method presented in Section 6.1.2.

Specifically, the expected *OOC* operation cost when the process is under the effect of state (i, j) at the beginning of a transition step can be, alternatively, computed as the sum of the following three terms:

(i) The cost per time unit if no assignable cause occurs within the interval, multiplied by the duration of the interval h_q, multiplied by the probability that no assignable cause occurs within the interval.

(ii) The cost per time unit in case an assignable cause occurs within the interval, multiplied by the expected time of the occurrence of this assignable cause t, multiplied by the probability that the assignable cause that occurs leads either to state (y, j) $\forall y \in (i,k]$ if it affects the process mean or (i,z) $\forall z \in (j,l]$, if it affects the standard deviation.

(iii) The *OOC* operation cost for state (y, j) or (i,z) being the initial state of the remainder of the interval $(h_q - t)$, in case an assignable cause occurs within the interval, multiplied by the probability that the assignable cause that occurs leads either to state (y, j) or (i,z), respectively.

In general, the expected *OOC* operation cost for every possible initial state when up to *m* and *r* assignable causes may occur, can be derived from the following expression:

$$\begin{aligned} K_{(i,j)}(h_q) = {} & M_{(i,j)} \cdot h_q \cdot \exp\left(-\left(v_{x,i} + v_{s,j}\right) \cdot h_q\right) + \\ & + \int_0^{h_q} \sum_{y=i+1}^{m} \left[\lambda_{x(i \to y)} \cdot \exp\left(-\left(v_{x,i} + v_{s,j}\right) \cdot t\right) \cdot \left(t \cdot M_{(i,j)} + K_{(y,j)}(h_q - t)\right) \right] dt + \\ & + \int_0^{h_q} \sum_{z=j+1}^{r} \left[\lambda_{s(j \to z)} \cdot \exp\left(-\left(v_{x,i} + v_{s,j}\right) \cdot t\right) \cdot \left(t \cdot M_{(i,j)} + K_{(i,z)}(h_q - t)\right) \right] dt \end{aligned} \tag{6.15}$$

6.2.3 Evaluation

The two aforementioned methods, i.e., the "depth-first search" and the "breadth-first search" method, were named based on the two popular search algorithms utilized for traversing a tree or a graph.

The "depth-first search" (*DFS*) algorithm follows one branch of the tree until the desirable node is found, whereas, the "breadth-first search" (*BFS*) algorithm scans each node of the first level by moving rightwardly, then the second level etc. until the desirable node is found.

Although, the choice between the two algorithms should depend on the data structure, in the worst case, which means generating all tree nodes, they have the same complexity. The implementation of the two algorithms in order to find every possible "path" for our process to move from one state (root node) to another state (desirable node) pertains to the worst case where every possible intermediate state (node) should be generated. Consequently, the "depth-first search" and the "breadth-first search" methods may be equivalently employed for the computation of the *OOC* operation cost.

However, in the special case where failures may also occur and the failure rate is correlated to the actual state of the process, the "breadth-first search" method is preferable to the "depth-first search" method because in the former the modification of the failure rate after each intermediate transition of the state of the process can be easily considered.

It should be mentioned that the two aforementioned methods could be also employed for the computation of the expected *OOC* operation cost for multivariate processes, by substituting λ_x with λ_{mv}, λ_s with λ_{cm}, v_x with v_{mv}, v_s with v_{cm}, m with m_{mv} and r with r_{cm}. The exact expression for the computation of $K_{(i,j)}\left(h_q\right)$ by utilizing the "breadth-first search" method can be found in Chapter 8.

7. *VP* $\overline{X}-s$ CONTROL SCHEME FOR PROCESSES SUBJECT TO MULTIPLE QUALITY SHIFTS AFFECTING BOTH LOCATION AND SCALE (*VP*$_3$)

7.1 Introduction

In this chapter, a fully adaptive control scheme for joint monitoring of the process location and scale is proposed, where multiple assignable causes affecting the mean and/or multiple assignable causes affecting the standard deviation of a process may occur independently, leading to a progressive deterioration of the process performance.

Obviously, by considering *m* different assignable causes that may affect the process mean and *r* assignable causes that may affect the standard deviation of the process, there are *m*+1 possible values of δ with $\delta \in \{0, \delta_1, \delta_2, ..., \delta_{m-1}, \delta_m\}$ $0 < \delta_1 < \delta_2 < ... < \delta_{m-1} < \delta_m$ and *r*+1 possible values of γ with $\gamma \in \{1, \gamma_1, \gamma_2, ..., \gamma_{r-1}, \gamma_r\}$ $1 < \gamma_1 < \gamma_2 < ... < \gamma_{r-1} < \gamma_r$.

This chapter is organized as follows. In Section 7.2 the mathematical model that describes the operation of the proposed scheme is described. Section 7.3 presents the formulation of the cost function. The optimization problem is presented in Section 7.4 and Section 7.5 provides a real example from the aviation industry. Finally, a numerical analysis is carried out in Section 7.6 and comparisons are performed against less adaptive control schemes.

This chapter gives the argumentation published in Tasias and Nenes (2016a).

7.2 Mathematical Model

The proposed *VP* $\overline{X}$ -*s* Shewhart type control scheme, denoted by VP_3, is an extension of the VP_1 control scheme, proposed in Chapter 4, where two independent assignable causes may occur, one affecting the process mean and one the standard deviation.

The decision making procedure of the control scheme is based on the standardized mean and the standard deviation of the collected sample at each sampling instance. Specifically: $a_t = 0$, if both the standardized sample mean z_t and the sample standard deviation s_t are below the respective threshold values (central zone), i.e., $z_t \le w_x$ and $s_t \le UWL_{s,q}$ (q=1 (q=2) if relaxed (tightened) parameters have been used). In such a case, the process continues its operation and relaxed parameters $\{n_1, h_1, w_x, k_{x,1}, w_s, k_{s,1}\}$ are used for the next sampling. Moreover, $a_t = 1$, if either the standardized sample mean, the sample standard deviation, or both, lie between the respective threshold and control limit coefficients (warning zone), but in none of the two charts is there an alarm signal. In particular: i) $w_x < z_t \le k_{x,q}$ and $s_t \le UCL_{s,q}$ or ii) $z_t \le w_x$ and $UWL_{s,q} < s_t \le UCL_{s,q}$. It should be noted that whenever $a_t = 1$ the decision for the process is to continue, but at the next sampling the tightened group of parameters $\{n_2, h_2, w_x, k_{x,2}, w_s, k_{s,2}\}$ should be used. Finally, $a_t = 2$, if the value of the chart statistic in at least one control chart outreaches the respective control limit (action zone), i.e., if either $z_t > k_{x,q}$ and/or $s_t > UCL_{s,q}$. Then, the process is stopped for investigation and either a false alarm is discovered, or the process was actually *OOC* and is perfectly restored to the *IC* condition. It should be noted that regardless of the control chart that issues the alarm, the investigation of the process reveals the effect of any possible assignable cause, that may have affected the mean or the standard deviation, and the process is assumed to restart its operation from the *IC* state. After this perfect restoration, relaxed parameters will be used, namely $\{n_1, h_1, w_x, k_{x,1}, w_s, k_{s,1}\}$.

The control policy for the control scheme is illustrated graphically in Figure 7-1.

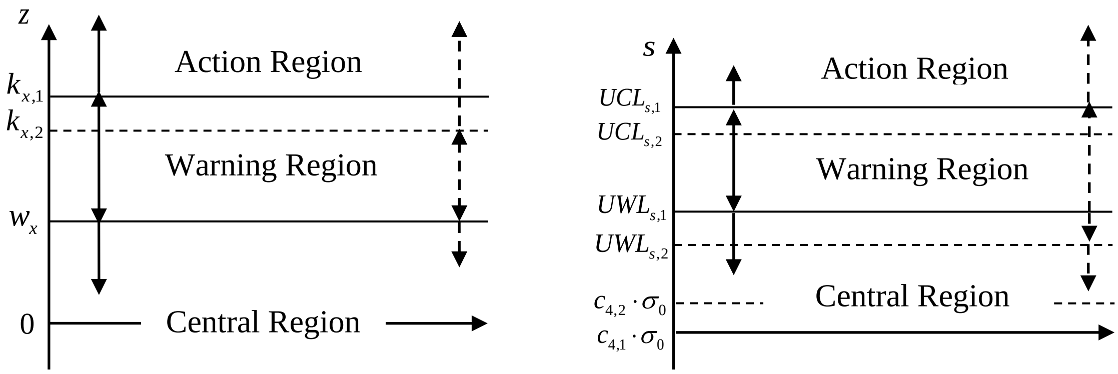


Figure 7-1: Regions of the VP_3 control scheme

The state of the process at any sampling instance *t* has the following possible values: (a) $Y_t = (0,0)$, when no assignable cause has occurred; (b) $Y_t = (i,0)$, when an assignable cause *i* (*i*=1,…,*m*) that affects the process mean has occurred, leading to a shift size $\delta = \delta_i$, but the standard deviation is equal to its *IC* value $(\sigma = \sigma_0)$; (c) $Y_t = (0, j)$, when an assignable cause *j* (*j*=1,..,*r*) that affects the standard deviation has occurred, leading to a shift size $\gamma = \gamma_j$, but the mean of the process is not affected $(\mu = \mu_0)$; (d) $Y_t = (i, j)$ *i*=1,...,*m* and *j*=1,...,*r*, when two assignable causes have contemporarily occurred and affect both the mean and the standard deviation of the process by shifting them to $\mu_i = \mu_0 + \delta_i \cdot \sigma_0$ and $\sigma_j = \gamma_j \cdot \sigma_0$, respectively.

Based on the values of the two components that constitute the three-dimensional state, the Markov chain has $(m+1)\times(r+1)\times 3$ possible (Y_t, a_t) states for each possible combination of $Y_t = \{\{(0,0),(0,1),...,(0,r)\}, \{(1,0),(1,1),...,(1,r)\},..., \{(m,0),(m,1),...,(m,r)\}\}$ and $a_t = 0,1,2$.

For ease of presentation, in order to keep the model at manageable levels and without any loss of generality, in many cases throughout this chapter, we assume that two assignable causes affecting the process mean and two assignable causes affecting the standard deviation of the process are possible to occur $(m = r = 2)$. This is not a restrictive assumption and is simply made to facilitate the exposition of the model operation. An extension to the general case, where any number of assignable causes could affect the process mean and standard deviation $(m, r \geq 2)$, will also be presented.

In case $m = r = 2$, the dimensions of the transition probability matrix P are $((m+1)\times(r+1)\times3)\times((m+1)\times(r+1)\times3) = 27\times27$ and the transition probability matrix P is shown in Figure 7-2.

In the general case, where m and r assignable causes are possible to occur, the transition probability matrix P can be obtained as shown in Figure 7-3.

		Y_t	k		0			0			0			1			1			1			2			2			2	
			l		0			1			2			0			1			2			0			1			2	
			α_t	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
	Y_{t-1} i	j	α_{t-1}																											
			0																											
	0	0	1																											
			2																											
			0																											
	0	1	1																											
			2																											
			0																											
	0	2	1																											
			2																											
			0																											
	1	0	1																											
			2																											
			0																											
P=	1	1	1																											
			2																											
			0																											
	1	2	1																											
			2																											
			0																											
	2	0	1																											
			2																											
			0																											
	2	1	1																											
			2																											
			0																											
	2	2	1																											
			2																											

Figure 7-2: Transition Probability Matrix of the VP_3 control scheme $\left(m = r = 2\right)$

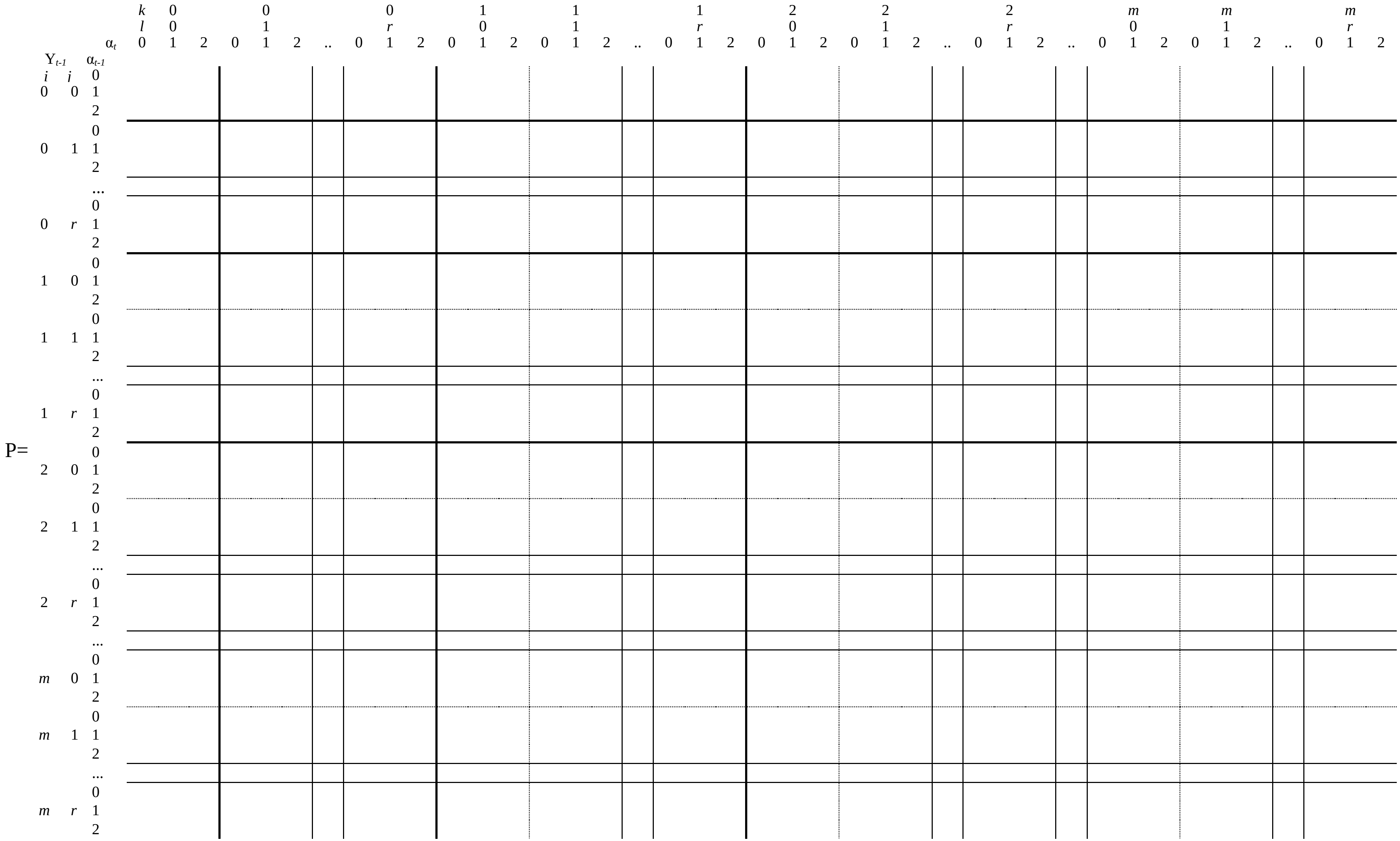


Figure 7-3: Transition Probability Matrix of the VP_3 control scheme

The transition probabilities may be computed as the product of multiplying the probability of the transition of the actual state of the process (equation (6.1) or (6.6)) with the probability of the decision a_t to be equal to 0,1 or 2. It is apparent that in case $(i > k) \cup (j > l)$ the transition probabilities are equal to zero.

$$Prob_{\substack{(i,j)a_{t-1}\\(k,l)a_t}}\left(h_q\right)=\begin{cases} p_{\substack{(i,j)\\(k,l)}}\left(h_q\right)\cdot\Phi\left(\dfrac{w_x-\delta_k\sqrt{n_q}}{\gamma_l}\right)\cdot P\left(\chi^2_{n_q\text{-}1}<\left(\dfrac{UWL_{s,q}}{\gamma_l\cdot\sigma_0}\right)^2\cdot\left(n_q\text{-}1\right)\right) & a_t=0\\ p_{\substack{(i,j)\\(k,l)}}\left(h_q\right)\cdot\left[\begin{array}{l}\Phi\left(\dfrac{k_{x,q}-\delta_k\sqrt{n_q}}{\gamma_l}\right)\cdot P\left(\chi^2_{n_q\text{-}1}<\left(\dfrac{UCL_{s,q}}{\gamma_l\cdot\sigma_0}\right)^2\cdot\left(n_q\text{-}1\right)\right)-\\ -\Phi\left(\dfrac{w_x-\delta_k\sqrt{n_q}}{\gamma_l}\right)\cdot P\left(\chi^2_{n_q\text{-}1}<\left(\dfrac{UWL_{s,q}}{\gamma_l\cdot\sigma_0}\right)^2\cdot\left(n_q\text{-}1\right)\right)\end{array}\right] & a_t=1\\ p_{\substack{(i,j)\\(k,l)}}\left(h_q\right)\cdot\left[1-\Phi\left(\dfrac{k_{x,q}-\delta_k\sqrt{n_q}}{\gamma_l}\right)\cdot P\left(\chi^2_{n_q\text{-}1}<\left(\dfrac{UCL_{s,q}}{\gamma_l\cdot\sigma_0}\right)^2\cdot\left(n_q\text{-}1\right)\right)\right] & a_t=2\end{cases} \tag{7.2}$$

Note that in case $a_{t-1}=2$, i.e., when the chart issues an alarm, the process always restarts its operation from the *IC* state and so the transition probabilities are computed from the following equation:

$$Prob_{\substack{(i,j)2\\(k,l)a_t}}\left(h_q\right)=Prob_{\substack{(0,0)0\\(k,l)a_t}}\left(h_1\right) \tag{7.3}$$

The steady-state probabilities, $\pi_{Y_t a_t}$, are computed by solving the following linear system:

$$\pi_{Y_t a_t}=\sum_{Y_{t-1}=(0,0)}^{(m,r)}\sum_{a_{t-1}=0}^{2}Prob_{\substack{Y_{t-1}a_{t-1}\\Y_t\ a_t}}\left(h_q\right)\cdot\pi_{Y_{t-1}a_{t-1}}\ \text{ and }\ \sum_{Y_t=(0,0)}^{(m,r)}\sum_{a_t=0}^{2}\pi_{Y_t a_t}=1 \tag{7.4}$$

7.3 The Economic-Statistical Design

From the computation of the steady-state probabilities of the process (equation (7.4)) and the computation of the mean cost of *OOC* operation (equation (6.14) or (6.15)), the average cost of a transition step, *EC*, and the average duration of a transition step, *ET*, can be evaluated.

The average cost of a transition step can be derived from the following expression:

$$EC = b + \sum_{k=0}^{m}\sum_{l=0}^{r}\pi_{(k,l)0}\cdot\left[c\cdot n_1 + K_{(k,l)}\left(h_1\right)\right] + \sum_{k=0}^{m}\sum_{l=0}^{r}\pi_{(k,l)1}\cdot\left[c\cdot n_2 + K_{(k,l)}\left(h_2\right)\right] + \\ + \sum_{k=0}^{m}\sum_{l=0}^{r}\pi_{(k,l)2}\cdot\left[c\cdot n_1 + K_{(0,0)}\left(h_1\right) + L_{(k,l)}\right] \qquad (7.10)$$

In the same sense, the expected duration of a transition step, *ET*, is computed by the following function:

$$ET = h_1\cdot\sum_{k=0}^{m}\sum_{l=0}^{r}\pi_{(k,l)0} + h_2\cdot\sum_{k=0}^{m}\sum_{l=0}^{r}\pi_{(k,l)1} + \sum_{k=0}^{m}\sum_{l=0}^{r}\pi_{(k,l)2}\cdot\left(h_1 + T_{(k,l)}\right) \qquad (7.11)$$

7.4 Optimization Problem

The optimization problem is formulated as follows:

$$\begin{gathered} \min_{DP_q} ECT \\ \text{s.t.} \quad h_1, h_2, n_1, n_2, w_x, k_{x,1}, k_{x,2}, w_s, k_{s,1}, k_{s,2} > 0 \\ h_2 \le h_1 \\ n_2 \ge n_1 \\ w_x \le k_{x,2} \le k_{x,1} \\ w_s \le k_{s,2} \le k_{s,1} \\ n_1, n_2 \in \mathbb{N}^+ \\ \alpha \le 0.02 \end{gathered} \qquad (7.12)$$

The minimization of *ECT* is achieved by means of a computer program developed in Fortran Power Station 4.0. The derivation of the optimum design parameters for the proposed control scheme for each examined case does not last more than three hours on an Intel i7-3520M dual-core processor at 2.9GHz.

7.5 An Illustrative Example

In this section, a real example from the aviation industry is employed in order to evaluate the performance of the proposed scheme. The Exhaust Gas Temperature

(*EGT*) is an important measure of a jet engine's health. *EGT* is the temperature of the gases as they leave the turbine unit and is computed through several thermocouples mounted in the perimeter of the exhaust duct. The early detection of an unreasonably high *EGT* would save maintenance costs, assure safety efficiency and prolong the lifespan of the engine.

An aircraft mechanic may perform the on-line monitoring of *EGT* through the proposed scheme by measuring the *EGT* of the engine in the run-up area. It is proved that the probability distribution of *EGT* can be well approximated by a normal one.

For our example, data are exploited from the quality assurance division of a fighter aircraft squadron. The *IC* values of the mean and standard deviation of *EGT*, defined in the engine's technical orders (TO's), are $\mu_0 = 1240$ ^{o}F and $\sigma_0 = 20$ ^{o}F, respectively.

We assume four possible assignable causes that result in an *OOC* condition of *EGT* at a specific throttle level (i.e., at maximum thrust): bleed air leaks, compressor blade tip clearance, throttle system misrigging and nozzle system misrigging. The first two result in an upward shift of the mean and the other two of the standard deviation of *EGT* $\left(m = r = 2\right)$. All the economic and statistical parameters of the process are estimated from observed data of the squadron and are presented in Table 7-1.

Table 7-1: Parameter set of the illustrative example for the VP_3 control scheme

Occurrence Rates (failures/100 flight hours)	**Magnitude of Shifts**	**Costs** (1000 $)			**Time Delays** (flight hours)
		Sampling	Out-of-control Operation	Removal of assignable causes	
$\lambda_{x(0\to1)} = 0.3$	$\delta_1 = 0.8$	$c = 0.1$	$M_{(0,1)} = 0.4$	$L_{(0,0)} = 0.5$	$T_{(0,0)} = 7.0$
$\lambda_{x(0\to2)} = 0.1$	$\delta_2 = 1.1$	$b = 0.3$	$M_{(0,2)} = 0.5$	$L_{(0,1)} = 1.8$	$T_{(0,1)} = 9.0$
$\lambda_{x(1\to2)} = 0.0$	$\gamma_1 = 1.3$		$M_{(1,0)} = 0.8$	$L_{(0,2)} = 2.3$	$T_{(0,2)} = 13.0$
$\lambda_{s(0\to1)} = 0.7$	$\gamma_2 = 1.6$		$M_{(1,1)} = 1.1$	$L_{(1,0)} = 2.4$	$T_{(1,0)} = 16.0$
$\lambda_{s(0\to2)} = 0.9$			$M_{(1,2)} = 1.2$	$L_{(1,1)} = 4.1$	$T_{(1,1)} = 25.0$
$\lambda_{s(1\to2)} = 0.001$			$M_{(2,0)} = 0.7$	$L_{(1,2)} = 4.5$	$T_{(1,2)} = 29.0$
			$M_{(2,1)} = 1.0$	$L_{(2,0)} = 2.8$	$T_{(2,0)} = 14.5$
			$M_{(2,2)} = 1.2$	$L_{(2,1)} = 4.4$	$T_{(2,1)} = 23.5$
				$L_{(2,2)} = 4.9$	$T_{(2,2)} = 27.5$

By solving the optimization problem for our example, we can define the optimum values of the design parameters to be: $h_1 = 12.0$, $h_2 = 0.1$ flight hours, $n_1 = n_2 = 2$, $w_x = 0.3$, $k_{x,1} = 1.2$, $k_{x,2} = 1.2$, $w_s = 1.5$, $k_{s,1} = 2.2$ and $k_{s,2} = 1.9$. These optimal design parameters dictate that *EGT* should be measured in the engine run-up twice ($n_1 = 2$) after twelve flight hours ($h_1 = 12.0$) and the standardized mean (standard deviation) of the two measures should be compared to $w_x = 0.3, k_{x,1} = 1.2$ $\left(UWL_{s,1} = 34, UCL_{s,1} = 42.4\right)$. If the scheme warns for the effect of an assignable cause, two additional *EGT* checks at maximum thrust $\left(n_2 = 2\right)$ should be conducted as soon as possible $\left(h_2 = 0.1\right)$ and then, the control limit coefficients would be $k_{x,2} = 1.2$ and $k_{s,2} = 1.9$ $\left(UCL_{s,2} = 38.8\right)$. Each time the scheme issues an alarm, the aircraft engine goes through an extensive inspection process.

The optimal *ECT* is equal to 56.089 $/flight hour and the *Type I* error equals 0.149. It should be noted that, unlike usual industry applications, in the specific example, because the operation at an *OOC* condition could jeopardize aircraft's safety, a somewhat higher value of the expected over-adjustments strengthens the aircraft mechanics' confidence in the quality control policy. For that reason, no statistical constraint has been put in the *Type I* error and, so, the proposed scheme has been only economically optimized.

7.6 Numerical Analysis

In this section a numerical investigation is performed in order to explore the economic and statistical performance of the proposed control scheme. The aforementioned approach for computing the optimum design parameters is applied to a benchmark of scenarios, which are defined by ten process $\left(\lambda, \delta, \gamma, T_{(0,0)}, T\right)$ and economic $\left(b, c, M, L_{(0,0)}, L\right)$ parameters that vary at two levels. Specifically, the ten parameters and their possible values are: $\lambda \in \left(0.01; 0.1\right)$ failures/hour, $\delta \in \left(0.5; 1.5\right)$, $\gamma \in \left(1.4; 2.0\right)$, $T_{(0,0)} \in \left(1; 6\right)$ minutes, $T \in \left(10; 30\right)$ minutes, $b \in \left(0; 5\right)$ \$, $c \in \left(1; 10\right)$ \$/sample unit, $M \in \left(100; 1000\right)$ \$/hour, $L_{(0,0)} \in \left(100; 200\right)$ \$ and $L \in \left(200; 400\right)$ \$. Because of the large number of possible scenarios, i.e., $2^{10} = 1024$, and for the sake of brevity, we have chosen to present a subset of 64 different runs, for every possible combination of the first six parameters $\left(\lambda, \delta, \gamma, T_{(0,0)}, T, b\right)$ and an arbitrary choice of the remaining four $\left(c, M, L_{(0,0)}, L\right)$.

Two assignable causes that affect the process mean (standard deviation) are possible to occur $\left(m = r = 2\right)$. Tables 7-2 and 7-3 present the benchmark of the process scenarios. It should be mentioned that $\lambda_{x(g \to g+1)} = \lambda_{s(g \to g+1)} = \lambda$ and $\lambda_{x(g \to g+2)} = \lambda_{s(g \to g+2)} = \lambda / 2$, where g=0,1,2. Moreover, $\delta_1 = \delta$, $\delta_2 = 1.5 \cdot \delta$ and $\gamma_1 = \gamma$, $\gamma_2 = 2 \cdot \gamma - 1$. As regards the time to search and remove an assignable cause $T_{(0,1)} = T_{(1,0)} = T / 2$, $T_{(0,2)} = T_{(2,0)} = T_{(1,1)} = T$, $T_{(2,1)} = T_{(1,2)} = 3 \cdot T / 2$, $T_{(2,2)} = 2 \cdot T$. Furthermore, $M_{(0,1)} = M_{(1,0)} = M$, $M_{(0,2)} = M_{(2,0)} = 1.5 \cdot M$ and $M_{(i,j)} = 0.75 \cdot (M_{(i,0)} + M_{(j,0)})$ if $i, j \geq 1$ and $L_{(0,1)} = L_{(1,0)} = L$, $L_{(0,2)} = L_{(2,0)} = L + 50$ and $L_{(i,j)} = 0.75 (L_{(i,0)} + L_{(j,0)})$ if $i, j \geq 1$.

The optimum design parameters, the expected quality control cost and the measures of statistical performance for each investigated scenario are presented in the Tables 7-4 and 7-5.

Table 7-2: Parameter sets of numerical examples 1-32 for the VP_3 control scheme

Case	λ	δ	γ	$T_{(0,0)}$	T	b	c	M	$L_{(0,0)}$	L
1	0.01	0.5	1.4	1	10	0	10	1000	200	400
2	0.1	0.5	1.4	1	10	0	10	100	100	200
3	0.01	1.5	1.4	1	10	0	1	1000	100	200
4	0.1	1.5	1.4	1	10	0	1	100	200	400
5	0.01	0.5	2.0	1	10	0	1	100	200	200
6	0.1	0.5	2.0	1	10	0	1	1000	100	400
7	0.01	1.5	2.0	1	10	0	10	100	100	400
8	0.1	1.5	2.0	1	10	0	10	1000	200	200
9	0.01	0.5	1.4	6	10	0	1	100	100	400
10	0.1	0.5	1.4	6	10	0	1	1000	200	200
11	0.01	1.5	1.4	6	10	0	10	100	200	200
12	0.1	1.5	1.4	6	10	0	10	1000	100	400
13	0.01	0.5	2.0	6	10	0	10	1000	100	200
14	0.1	0.5	2.0	6	10	0	10	100	200	400
15	0.01	1.5	2.0	6	10	0	1	1000	200	400
16	0.1	1.5	2.0	6	10	0	1	100	100	200
17	0.01	0.5	1.4	1	30	0	10	1000	100	200
18	0.1	0.5	1.4	1	30	0	10	100	200	400
19	0.01	1.5	1.4	1	30	0	1	1000	200	400
20	0.1	1.5	1.4	1	30	0	1	100	100	200
21	0.01	0.5	2.0	1	30	0	1	100	100	400
22	0.1	0.5	2.0	1	30	0	1	1000	200	200
23	0.01	1.5	2.0	1	30	0	10	100	200	200
24	0.1	1.5	2.0	1	30	0	10	1000	100	400
25	0.01	0.5	1.4	6	30	0	1	100	200	200
26	0.1	0.5	1.4	6	30	0	1	1000	100	400
27	0.01	1.5	1.4	6	30	0	10	100	100	400
28	0.1	1.5	1.4	6	30	0	10	1000	200	200
29	0.01	0.5	2.0	6	30	0	10	1000	200	400
30	0.1	0.5	2.0	6	30	0	10	100	100	200
31	0.01	1.5	2.0	6	30	0	1	1000	100	200
32	0.1	1.5	2.0	6	30	0	1	100	200	400

Table 7-3: Parameter sets of numerical examples 33-64 for the VP_3 control scheme

Case	λ	δ	γ	$T_{(0,0)}$	T	b	c	M	$L_{(0,0)}$	L
33	0.01	0.5	1.4	1	10	5	1	100	200	400
34	0.1	0.5	1.4	1	10	5	1	1000	100	200
35	0.01	1.5	1.4	1	10	5	10	100	100	200
36	0.1	1.5	1.4	1	10	5	10	1000	200	400
37	0.01	0.5	2.0	1	10	5	10	1000	200	200
38	0.1	0.5	2.0	1	10	5	10	100	100	400
39	0.01	1.5	2.0	1	10	5	1	1000	100	400
40	0.1	1.5	2.0	1	10	5	1	100	200	200
41	0.01	0.5	1.4	6	10	5	10	1000	100	400
42	0.1	0.5	1.4	6	10	5	10	100	200	200
43	0.01	1.5	1.4	6	10	5	1	1000	200	200
44	0.1	1.5	1.4	6	10	5	1	100	100	400
45	0.01	0.5	2.0	6	10	5	1	100	100	400
46	0.1	0.5	2.0	6	10	5	1	1000	200	200
47	0.01	1.5	2.0	6	10	5	10	100	200	400
48	0.1	1.5	2.0	6	10	5	10	1000	100	200
49	0.01	0.5	1.4	1	30	5	1	100	100	200
50	0.1	0.5	1.4	1	30	5	1	1000	200	400
51	0.01	1.5	1.4	1	30	5	10	100	200	400
52	0.1	1.5	1.4	1	30	5	10	1000	100	200
53	0.01	0.5	2.0	1	30	5	10	1000	100	400
54	0.1	0.5	2.0	1	30	5	10	100	200	200
55	0.01	1.5	2.0	1	30	5	1	1000	200	200
56	0.1	1.5	2.0	1	30	5	1	100	100	400
57	0.01	0.5	1.4	6	30	5	10	1000	200	200
58	0.1	0.5	1.4	6	30	5	10	100	100	400
59	0.01	1.5	1.4	6	30	5	1	1000	100	400
60	0.1	1.5	1.4	6	30	5	1	100	200	200
61	0.01	0.5	2.0	6	30	5	1	100	200	400
62	0.1	0.5	2.0	6	30	5	1	1000	100	200
63	0.01	1.5	2.0	6	30	5	10	100	100	200
64	0.1	1.5	2.0	6	30	5	10	1000	200	400

Table 7-4: Economic-Statistical design for numerical examples 1-32: optimal control policy, cost and related statistical measures for the VP_3 control scheme

Optimum Design Parameters												Statistical Measures						
Case	h_1	h_2	n_1	n_2	w_x	$k_{x,1}$	$k_{x,2}$	w_s	$k_{s,1}$	$k_{s,2}$	ECT_{VP3}	α	$1-\beta$	*ANOF*	ARL_0	*WARL*	*ATC*	*EATR*
1	0.7	0.1	2	8	0.9	2.7	1.7	1.7	3.6	2.1	139.033	0.0198	0.2030	0.0313	50.51	4.926	36.58	3.25
2	0.9	0.3	2	2	0.9	2.5	1.7	1.4	3.5	2.9	134.628	0.0200	0.1699	0.0106	50.00	5.887	7.19	3.86
3	0.3	0.1	3	11	1.8	3.0	2.9	1.1	3.5	2.0	40.374	0.0077	0.3399	0.0264	129.87	2.942	36.11	2.78
4	0.1	0.1	2	2	2.1	3.5	2.5	2.6	4.7	3.2	127.011	0.0009	0.0876	0.0057	1111.11	11.410	5.30	1.97
5	1.5	0.1	4	14	0.9	3.5	2.1	2.1	4.1	3.6	18.153	0.0042	0.3331	0.0031	238.10	3.002	35.89	2.56
6	0.1	0.1	2	11	0.7	2.9	1.7	2.0	4.0	3.0	207.338	0.0143	0.3623	0.1096	69.93	2.760	4.27	0.94
7	3.0	0.1	2	3	1.3	2.4	2.1	1.4	3.1	2.3	31.273	0.0198	0.4725	0.0068	50.51	2.116	36.93	3.60
8	0.3	0.1	2	3	1.2	2.4	2.1	1.2	3.1	2.4	246.527	0.0198	0.4836	0.0605	50.51	2.068	4.04	0.71
9	0.1	0.1	2	3	1.8	3.0	1.8	3.6	4.0	3.7	21.103	0.0039	0.4823	0.0182	256.41	20.736	70.83	37.50
10	0.1	0.1	3	20	1.1	3.2	2.1	1.5	3.8	2.3	113.906	0.0073	0.2941	0.0327	136.99	3.400	7.37	4.04
11	2.5	0.1	2	3	1.3	2.6	2.2	1.0	3.2	2.0	32.579	0.0199	0.2733	0.0080	50.25	3.659	40.43	7.10
12	0.1	0.1	2	2	1.3	2.5	2.0	1.0	3.2	2.4	218.647	0.0200	0.2435	0.0874	50.00	4.106	7.51	4.18
13	0.5	0.1	2	2	0.2	2.6	1.8	1.8	3.9	3.8	90.466	0.0200	0.1675	0.0445	50.00	5.968	45.58	12.25
14	0.8	0.3	2	2	0.6	2.5	1.7	1.7	3.7	3.4	147.302	0.0198	0.2307	0.0137	50.51	4.335	6.52	3.19
15	0.1	0.1	2	9	2.2	3.6	3.0	2.8	5.2	3.1	22.034	0.0003	0.3043	0.0016	3333.33	3.286	67.34	34.01
16	0.1	0.1	2	4	2.1	3.3	2.4	3.1	4.6	3.4	45.542	0.0011	0.2737	0.0047	909.09	3.654	7.42	4.09
17	0.6	0.1	2	6	0.9	2.6	1.7	1.5	3.9	2.2	129.056	0.0199	0.1820	0.0369	50.25	5.494	36.90	3.57
18	1.1	0.3	2	2	1.0	2.4	1.7	1.8	3.5	2.7	160.308	0.0200	0.1791	0.0071	50.00	5.582	8.23	4.90
19	0.3	0.1	3	16	1.9	3.3	3.2	1.3	4.1	2.2	47.384	0.0032	0.3226	0.0107	312.50	3.100	36.32	2.99
20	0.2	0.1	2	5	1.7	3.0	2.5	1.6	4.0	2.1	77.995	0.0060	0.2231	0.0215	166.67	4.483	4.75	1.42
21	1.3	0.1	3	10	0.8	3.2	1.8	1.9	4.0	3.3	23.170	0.0097	0.3132	0.0086	103.09	3.193	36.18	2.85
22	0.1	0.1	3	16	1.0	3.2	1.9	2.4	3.9	3.3	157.778	0.0060	0.3582	0.0437	166.67	2.792	4.51	1.18
23	3.1	0.1	2	3	1.3	2.4	2.1	1.4	3.1	2.3	26.229	0.0198	0.4730	0.0065	50.51	2.114	37.27	3.94
24	0.3	0.1	2	3	1.2	2.4	2.1	1.1	3.2	2.3	275.008	0.0200	0.4849	0.0581	50.00	2.062	4.27	0.94
25	0.2	0.1	2	15	1.8	3.7	2.1	2.8	4.2	2.9	18.537	0.0019	0.0845	0.0060	526.32	11.833	53.96	20.63
26	0.1	0.1	2	13	0.8	2.7	1.8	1.2	3.9	2.1	131.626	0.0200	0.3064	0.0867	50.00	3.264	7.57	4.24
27	2.3	0.1	2	3	1.4	2.5	2.2	1.1	3.2	2.0	36.647	0.0199	0.2727	0.0084	50.25	3.667	40.63	7.30
28	0.1	0.1	2	2	1.4	2.5	1.9	1.4	3.2	2.2	193.797	0.0198	0.2434	0.0841	50.51	4.109	7.75	4.42
29	0.6	0.1	2	4	0.7	2.6	1.6	1.9	4.0	2.9	98.590	0.0200	0.2291	0.0329	50.00	4.366	42.57	9.24
30	0.7	0.1	2	2	0.7	2.4	1.7	1.9	3.8	3.4	105.447	0.0200	0.2182	0.0167	50.00	4.584	6.58	3.25
31	0.1	0.1	2	7	2.1	3.3	2.7	2.7	4.7	3.0	18.728	0.0010	0.3195	0.0047	1000.00	3.130	67.53	34.20
32	0.1	0.1	2	2	2.1	3.7	2.5	3.3	5.3	3.9	70.782	0.0003	0.1768	0.0014	3333.33	5.658	8.07	4.74

Table 7-5: Economic-Statistical design for numerical examples 33-64: optimal control policy, cost and related statistical measures for the VP_3 control scheme

Optimum Design Parameters												*Statistical Measures*						
Case	h_1	h_2	n_1	n_2	w_x	$k_{x,1}$	$k_{x,2}$	w_s	$k_{s,1}$	$k_{s,2}$	ECT_{VP3}	α	$1\text{-}\beta$	*ANOF*	ARL_0	*WARL*	*ATC*	*EATR*
33	3.3	0.1	13	25	1.2	2.6	2.1	1.4	2.9	2.4	29.292	0.0117	0.4800	0.0037	85.47	2.083	36.56	3.23
34	0.3	0.1	12	23	1.0	2.4	2.0	1.2	2.7	2.2	217.781	0.0199	0.5196	0.0630	50.25	1.925	3.98	0.65
35	3.0	0.1	2	4	1.3	2.5	2.4	1.1	3.2	1.9	35.401	0.0200	0.3000	0.0068	50.00	3.334	39.23	5.90
36	0.3	0.1	2	4	1.3	2.5	2.3	0.8	3.3	2.0	383.670	0.0198	0.3130	0.0590	50.51	3.195	4.30	0.97
37	1.3	0.1	3	8	0.7	2.6	1.6	1.8	3.5	2.9	114.640	0.0200	0.3411	0.0185	50.00	2.932	35.82	2.49
38	1.3	0.3	2	2	1.2	2.3	1.5	2.4	3.7	2.9	165.593	0.0200	0.2669	0.0065	50.00	3.747	7.04	3.71
39	0.7	0.1	5	9	1.8	3.0	2.9	1.5	3.3	2.7	44.415	0.0037	0.6745	0.0056	270.27	1.483	34.82	1.49
40	0.8	0.1	5	8	1.9	3.0	2.9	1.7	3.4	2.7	87.086	0.0032	0.6766	0.0031	312.50	1.478	4.13	0.80
41	0.8	0.1	2	8	0.9	2.8	1.7	1.4	3.8	2.1	143.750	0.0199	0.2091	0.0285	50.25	4.782	36.58	3.25
42	1.0	0.3	2	2	0.9	2.4	1.7	1.7	3.6	2.9	141.925	0.0200	0.1747	0.0089	50.00	5.723	7.50	4.17
43	0.8	0.1	8	23	2.2	3.3	3.2	1.1	3.1	2.2	51.562	0.0051	0.5448	0.0068	196.08	1.835	34.96	1.63
44	0.9	0.3	3	6	1.5	2.5	2.4	1.0	2.9	1.9	140.347	0.0190	0.4354	0.0146	52.63	2.297	4.80	1.47
45	3.0	0.1	9	16	0.9	2.3	1.7	2.1	3.3	3.2	26.188	0.0183	0.5692	0.0066	54.64	1.757	35.98	2.65
46	0.3	0.1	10	20	0.9	2.4	1.9	2.1	3.3	3.2	198.087	0.0133	0.5988	0.0409	75.19	1.670	3.93	0.60
47	3.5	0.1	2	3	1.3	2.4	2.1	1.4	3.1	2.3	33.733	0.0198	0.4747	0.0057	50.51	2.107	37.42	4.09
48	0.3	0.1	2	3	1.2	2.3	2.2	1.3	3.2	2.4	258.250	0.0199	0.4831	0.0600	50.25	2.070	4.04	0.71
49	2.8	0.1	10	19	1.0	2.5	1.9	1.3	2.7	2.2	22.204	0.0198	0.4489	0.0076	50.51	2.228	37.99	4.66
50	0.1	0.1	3	20	1.1	3.2	2.1	1.5	3.9	2.3	161.092	0.0072	0.2934	0.0310	138.89	3.409	7.60	4.27
51	2.9	0.1	2	4	1.4	2.5	2.3	1.1	3.2	1.9	39.721	0.0200	0.2988	0.0066	50.00	3.346	40.86	7.53
52	0.1	0.1	2	2	1.3	2.5	2.0	1.0	3.2	2.4	209.252	0.0200	0.2435	0.0848	50.00	4.106	7.75	4.42
53	0.7	0.1	2	5	0.7	2.6	1.6	1.9	4.1	2.8	103.959	0.0199	0.2522	0.0290	50.25	3.965	41.61	8.28
54	0.9	0.3	2	2	0.6	2.5	1.7	1.6	3.8	3.3	113.565	0.0200	0.2346	0.0116	50.00	4.262	6.93	3.60
55	0.5	0.1	4	9	1.8	3.2	3.1	1.6	3.8	2.9	34.123	0.0017	0.5686	0.0029	588.24	1.759	41.45	8.12
56	0.1	0.1	2	2	2.0	3.5	2.4	3.2	4.9	3.7	94.446	0.0007	0.2062	0.0029	1428.57	4.850	7.90	4.57
57	0.5	0.1	2	5	0.9	2.6	1.7	1.6	3.8	2.2	122.267	0.0199	0.1669	0.0364	50.25	5.992	44.23	10.90
58	0.8	0.3	2	2	0.9	2.4	1.7	1.7	3.6	2.9	150.266	0.0200	0.1682	0.0103	50.00	5.946	8.20	4.87
59	0.5	0.1	4	12	1.7	2.9	2.8	0.9	3.1	1.9	48.495	0.0120	0.4283	0.0224	83.33	2.335	42.44	9.11
60	0.1	0.1	2	5	2.1	3.3	2.6	2.3	4.5	2.6	72.305	0.0015	0.1309	0.0058	666.67	7.638	8.48	5.15
61	2.5	0.1	7	17	0.9	3.0	2.0	2.1	3.7	3.6	25.436	0.0061	0.4471	0.0025	163.93	2.236	37.72	4.39
62	0.1	0.1	2	11	0.7	2.9	1.7	1.9	4.0	2.9	115.165	0.0147	0.3660	0.0645	68.03	2.732	7.46	4.13
63	3.2	0.1	2	3	1.3	2.4	2.1	1.4	3.1	2.3	26.648	0.0198	0.4734	0.0061	50.51	2.112	38.53	5.20
64	0.1	0.1	2	2	1.5	2.7	1.9	2.3	3.7	2.9	217.429	0.0087	0.3635	0.0382	114.94	2.751	7.47	4.14

The examination of the economic and statistical results presented above leads to the following useful conclusions.

It becomes immediately evident that in the majority of the examined cases (87.5 %), the value of the tightened sampling interval h_2 equals its minimum allowable value, which is set to 0.1 time units, and, so, a second sample should be collected as soon as possible whenever a warning is issued by the scheme.

Obviously, larger values of *OOC* operation costs *M*, lead to larger values of *ECT*. Another conclusion is that for high occurrence rates λ, the scheme indicates more frequent sampling, i.e., smaller values of h_1. The exact same conclusion, for more frequent sampling, applies, also, to larger values of M, in order for the *OOC* operation period to be minimized. Furthermore, the lower the effect of the assignable causes to the mean and/or the standard deviation, δ and γ, respectively, the larger the values of *ECT* and the values of *Type I* errors α.

The aforementioned results for larger values of *ECT* and *Type I* errors in case of high operation costs *M* and low assignable causes' effects, δ and γ, get even worse in case of high occurrence rates. The logical explanation of this conclusion is that in such cases, with lower values of δ, γ and higher values of λ, it is more difficult, or equivalently more "expensive", for the scheme to identify an alarm and to distinguish whether this alarm is a true or a false one. The proposed scheme counterbalances these effects by indicating more frequent sampling and/or "tighter" threshold and control limits.

In order to evaluate the cost savings associated with monitoring a process in presence of multiple assignable causes, an economic comparison with the respective control scheme with design parameters optimized for monitoring one assignable cause for the mean and one for the standard deviation, i.e., VP_1 control scheme, is made.

For ease of reading, the expected cost per time unit for the VP_3 control scheme is denoted by *ECTm*, whereas, in case a single assignable cause may affect the mean and another one the standard deviation, the respective cost is denoted by *ECTs*.

The effect of each of the two assignable causes (δ', γ') is considered to be equal to the average of the effect of the two assignable causes affecting the mean and the two causes affecting the standard deviation, considered for the numerical investigation of the proposed scheme, i.e., $\delta' = \langle \delta \rangle$ $(\delta = (\delta_1, \delta_2))$ and $\gamma' = \langle \gamma \rangle$ $(\gamma = (\gamma_1, \gamma_2))$. Then, the VP_1 control scheme is optimized and the optimum values of every parameter are evaluated for each of the examined cases. The aforementioned values of every design parameter are utilized in the proposed model (VP_3) for the computation of the expected cost per time unit *ECTs* and this cost is compared to *ECTm*.

It should be mentioned that *ECTm* refers to the economic outcome of the proposed scheme optimized only economically, without the *Type I* error constraint. The reasoning is that the design of the scheme after the consideration of only one assignable cause for the mean and one for the standard deviation does not necessarily result in *Type I* error lower than 0.02 for the process. It is apparent that imposing no constraint to the *Type I* error, results in *ECTm* values at most equal or even lower compared to the minimum *ECT* computed with the statistical constraint $(a \leq 0.02)$ which has been presented in Tables 7-4 and 7-5 for each case. The results of the comparison are presented in Table 7-6.

Table 7-6: Economic comparison between the VP_3 control scheme and the VP_1 control scheme. Numerical examples 1-32

	VP_1										VP_3		$\frac{ECTs-ECTm}{ECTs}$ (%)
Case	h_1	h_2	n_1	n_2	w_x	$k_{x,1}$	$k_{x,2}$	w_s	$k_{s,1}$	$k_{s,2}$	*ECTs*	*ECTm*	
1	2.2	0.7	4	4	0.4	1.2	1.1	0.5	1.6	1.6	126.119	123.045	2.44
2	3.0	0.4	2	9	0.1	0.1	0.1	0.1	0.1	0.1	126.387	119.363	5.56
3	0.4	0.1	3	12	1.9	2.9	2.3	1.3	3.5	2.3	42.990	40.374	6.09
4	0.9	0.1	4	10	1.9	3.0	2.9	1.1	3.4	2.3	135.130	127.011	6.01
5	1.8	0.1	4	13	0.9	3.6	2.2	2.2	4.3	4.0	18.595	18.153	2.38
6	0.4	0.1	9	12	0.7	2.0	1.7	1.9	3.2	2.3	237.440	207.338	12.68
7	3.8	0.1	2	2	1.3	2.2	2.0	1.4	2.6	2.4	31.608	30.956	2.06
8	0.5	0.2	3	3	1.3	2.3	2.3	1.0	2.6	2.3	262.175	246.168	6.11
9	2.3	0.1	6	13	0.9	2.8	2.1	1.3	3.1	2.5	25.586	21.103	17.52
10	0.4	0.1	12	18	1.0	2.3	2.0	1.3	2.6	2.3	176.346	113.906	35.41
11	3.9	0.1	2	3	1.3	2.1	2.1	1.0	2.3	1.6	32.233	31.494	2.29
12	0.6	0.3	3	3	1.0	1.4	1.4	0.1	1.1	0.8	294.488	215.541	26.81
13	1.6	0.8	2	2	0.1	0.9	0.8	0.1	1.8	1.8	84.612	80.566	4.78
14	3.3	1.4	2	2	0.1	1.2	0.9	0.1	1.8	1.8	163.032	147.295	9.65
15	0.4	0.1	3	6	1.9	3.2	2.3	1.9	4.1	2.3	30.032	22.034	26.63
16	0.7	0.1	3	6	1.6	2.9	2.3	1.6	3.8	2.3	70.263	45.542	35.18
17	1.9	0.8	2	2	0.1	0.6	0.6	0.1	0.8	0.8	96.770	95.502	1.31
18	4.2	1.3	2	2	0.1	0.6	0.6	0.1	0.8	0.8	172.721	156.289	9.51
19	0.4	0.1	3	12	1.9	3.2	2.3	1.3	4.1	2.3	50.305	47.384	5.81
20	0.8	0.1	4	8	1.8	2.8	2.7	0.8	3.0	2.1	81.675	77.995	4.51
21	1.9	0.1	4	11	0.8	3.1	2.0	2.1	3.9	3.8	23.638	23.170	1.98
22	0.4	0.1	9	15	0.7	2.6	2.0	1.9	3.5	2.3	182.564	157.778	13.58
23	3.7	0.1	2	3	1.4	2.6	2.4	1.5	3.3	2.6	26.973	26.229	2.76
24	0.5	0.2	3	3	1.3	2.0	2.0	1.0	2.3	2.0	284.976	269.820	5.32
25	2.2	0.1	6	15	1.0	3.2	2.3	1.3	3.8	2.3	20.673	18.537	10.33
26	0.4	0.1	12	15	1.0	2.0	1.7	1.0	2.3	2.3	198.682	131.626	33.75
27	4.6	2.6	2	2	0.9	1.5	1.4	0.1	1.5	1.2	34.296	33.824	1.38
28	0.5	0.2	3	3	1.3	2.0	2.0	0.7	1.7	1.4	253.086	193.714	23.46
29	1.6	0.1	3	4	0.6	1.6	1.2	1.6	2.8	2.8	103.383	95.169	7.95
30	2.5	1.2	2	2	0.1	0.7	0.5	0.1	1.1	1.1	109.481	101.596	7.20
31	0.4	0.1	3	6	1.9	3.2	2.3	1.9	3.8	2.3	24.516	18.728	23.61
32	0.7	0.1	3	6	1.9	3.2	2.3	1.9	4.1	2.3	109.199	70.782	35.18

Table 7-7: Economic comparison between the VP_3 control scheme and the VP_1 control scheme. Numerical examples 33-64

	VP_1										VP_3		$\frac{ECTs-ECTm}{ECTs}$
Case	h_1	h_2	n_1	n_2	w_x	$k_{x,1}$	$k_{x,2}$	w_s	$k_{s,1}$	$k_{s,2}$	*ECTs*	*ECTm*	(%)
33	4.3	0.1	15	24	1.3	2.6	2.3	1.6	3.2	2.3	30.200	29.292	3.01
34	0.4	0.1	12	18	1.0	2.0	2.0	1.3	2.3	2.3	222.644	214.455	3.68
35	5.4	3.3	2	2	0.7	1.3	1.3	0.1	1.2	0.9	31.199	31.164	0.11
36	0.6	0.3	3	3	1.3	2.0	1.7	0.4	1.7	1.4	381.115	367.687	3.52
37	2.0	0.1	4	4	0.6	1.4	1.2	1.6	2.6	2.6	110.018	107.057	2.69
38	3.9	0.9	2	15	0.1	0.1	0.1	0.1	0.1	0.1	174.685	160.516	8.11
39	0.9	0.1	5	8	2.0	3.1	3.0	1.7	3.4	3.0	46.177	44.415	3.82
40	1.1	0.1	5	7	2.0	3.1	3.0	1.7	3.5	3.0	88.346	87.086	1.43
41	2.3	0.8	2	12	0.1	0.1	0.1	0.1	0.1	0.1	106.152	104.949	1.13
42	4.2	0.7	2	9	0.1	0.1	0.1	0.1	0.1	0.1	149.075	130.702	12.32
43	1.0	0.1	9	18	1.9	3.5	2.3	1.3	3.2	2.2	54.409	51.562	5.23
44	1.5	0.1	6	11	2.0	2.8	2.7	1.1	2.4	1.9	142.422	140.347	1.46
45	3.5	0.1	8	15	0.9	2.5	1.9	2.3	3.6	3.5	26.675	26.188	1.83
46	0.4	0.1	12	18	1.0	2.3	2.0	1.9	3.5	2.3	204.910	198.087	3.33
47	5.4	0.2	3	4	1.6	2.5	2.4	1.5	2.9	2.4	34.599	33.718	2.55
48	0.5	0.2	3	3	1.6	2.0	2.0	1.3	2.0	2.0	262.978	253.079	3.76
49	4.0	0.1	12	15	1.0	2.3	2.0	1.3	2.6	2.3	23.041	22.160	3.82
50	0.4	0.1	15	21	1.3	2.3	2.0	1.6	2.6	2.3	222.080	161.092	27.46
51	5.8	1.9	3	4	1.4	2.0	2.0	0.7	1.8	1.3	39.290	38.296	2.53
52	0.6	0.3	3	3	1.3	1.4	1.4	0.4	0.8	0.8	249.623	205.752	17.57
53	2.1	1.3	3	3	0.1	0.7	0.6	0.1	1.5	1.5	95.201	91.192	4.21
54	3.4	1.7	3	3	0.1	1.0	0.8	0.1	1.6	1.6	124.675	110.879	11.07
55	0.9	0.1	5	9	2.1	3.3	3.2	1.7	3.7	3.2	37.018	34.123	7.82
56	1.3	0.1	5	6	2.0	2.9	2.8	1.7	3.1	2.7	121.754	94.446	22.43
57	2.6	1.2	4	4	0.1	1.0	0.9	0.1	1.3	1.3	116.917	110.928	5.12
58	3.6	0.9	2	6	0.1	0.1	0.1	0.1	0.1	0.1	160.491	146.471	8.74
59	1.0	0.1	6	15	1.9	2.9	2.3	1.0	2.9	2.3	54.089	48.495	10.34
60	1.3	0.1	7	15	2.2	3.1	3.0	1.1	2.9	2.2	85.360	72.305	15.29
61	3.6	0.1	9	18	1.0	2.9	2.2	2.4	3.8	3.8	26.198	25.436	2.91
62	0.4	0.1	9	12	0.7	2.0	1.7	1.9	3.2	2.3	154.293	115.165	25.36
63	5.3	0.1	3	3	1.5	2.2	2.1	1.5	2.2	2.1	27.469	26.249	4.44
64	0.5	0.2	3	3	1.6	2.3	2.0	1.3	2.6	2.0	264.974	217.429	17.94

From the economic results given in Tables 7-6 and 7-7, it becomes evident that by regarding two assignable causes, one for the mean and one for the standard deviation, the process is imposed with a significant additional cost. Specifically, the average cost saving from the implementation of the proposed model is 9.91%. The value of the cost saving is greater in cases where M and λ have large values and the relative difference between the assignable causes is greater, i.e., $\delta = 1.5$, $\gamma = 2.0$.

8. *VP* MAINTENANCE AND QUALITY CONTROL SCHEME FOR PROCESSES SUBJECT TO FAILURES AND MULTIPLE QUALITY SHIFTS AFFECTING BOTH LOCATION AND SCALE (*VP*$_4$)

8.1 Introduction

In this chapter we propose an integrated *SPC* and maintenance model by employing a *VP* control scheme for joint monitoring of mean and standard deviation of processes where multiple independent assignable causes, affecting both the mean and standard deviation, are possible to occur. An additional feature of the monitored processes is that, apart from the quality shifts, failures may, also, occur any time within the production cycle.

Consequently, the process is subject to a multiple assignable cause mechanism, which consists of m quality shifts of the mean and r of the standard deviation $(m \geq 1, r \geq 1)$, plus a failure. The time to the occurrence of each quality shift and failure follows a non-negative exponential distribution.

In the proposed model, each quality shift not only increases the probability of an inferior shift to occur, but it also scales-up the failure rate, in proportion to the magnitude of the shift to the process. Due to the dependence of the failure rate on the process state, the former is denoted by $\lambda_{Fx|i}$ $\left(\lambda_{Fs|j}\right)$ when the process mean (standard deviation) is under the effect of i (j). As already mentioned, $\lambda_{Fx|k} > \lambda_{Fx|i}$ for $k > i$ $\left(\lambda_{Fs|l} > \lambda_{Fs|j}\right.$ for $\left. l > j\right)$. The overall failure rate when the process operates under the effect of any state (i, j) equals the sum of the failure rates $\left(\lambda_{F|(i,j)} = \lambda_{Fx|i} + \lambda_{Fs|j}\right)$.

Furthermore, a distinctive difference between quality shifts and failures is that *OOC* operation is not directly observable, in contrast to the failure state, which ceases the process operation.

After a confirmed quality shift upon inspection, the process is perfectly restored to the *IC* state, which consists a *PM* action. On the other hand, each time a failure occurs the process is again perfectly restored to the *IC* state, but, this consists a *CM* action.

It should be noted that the detection of a quality shift is usually preferable to precede a failure. The aforementioned assumption can be justified by a considerably lower cost of a *PM* action $\left(L_{(i,j)}\right)$ compared to the cost of a *CM* action, denoted by L_F $\left(L_F > L_{(k,l)} > L_{(i,j)} > 0 \text{ for } k > i > 0,\ l > j > 0\right)$. Furthermore, the time delays are higher in case of a failure T_F than an *OOC* signal and ($T_F > T_{(k,l)} > T_{(i,j)} > 0$ for $k > i > 0,\ l > j > 0$).

This chapter is structured as follows. In Section 8.2 we develop the mathematical model and in Section 8.3 the definition of the expected cycle time length and cost is presented. The expected availability is computed in Section 8.4, while Section 8.5 extends the model to cases where an imperfect restoration of the process to the *IC* state can occur. Section 8.6 shows the formulation of the optimization problem and the generation method utilized to tackle it. Finally, in Section 8.7, a numerical investigation is presented.

It should be mentioned that this chapter is based on a working paper of Tasias and Nenes (2016b).

8.2 Mathematical Model

The process is monitored through a *VP* $\overline{X}$ -*s* control scheme, denoted by VP_4. At each sampling instance, z_t and s_t of the collected sample are computed and one of the following three decisions, which summarize the scheme's operation, is made:

(a) If $z_t > k_{x,q}$ and/or $s_t > UCL_{s,q}$ $\left(a_t = 2\right)$, there is an out-of-control signal and the process is halted for investigation which reveals either a false alarm and no action is taken or an *OOC* operation; the latter dictates a *PM* action which restores the process either perfectly to the *IC* state or partially to a non-inferior state with known probabilities (see Section 3.4.2). Without loss of generality, we utilize the relaxed scheme, i.e., $\left\{n_1, h_1, w_x, k_{x,1}, w_s, k_{s,1}\right\}$, after a *PM* action;

(b) If $w_x < z_t \le k_{x,q}$ and/or $UWL_{s,q} < s_t \le UCL_{s,q}$ $(a_t = 1)$, there is no interruption, but there is a warning and the tightened set of parameters should be used $\{n_2, h_2, w_x, k_{x,2}, w_s, k_{s,2}\}$;

(c) If $z_t \le w_x$ and $s_t \le UWL_{s,q}$ $(a_t = 0)$, the process continues its operation and relaxed parameters are utilized.

As already mentioned, if at any time the process operation is ceased because of a failure, the process will be restored to the *IC* state through a *CM* action.

The control policy for the two control charts is illustrated graphically in Figure 8-1.

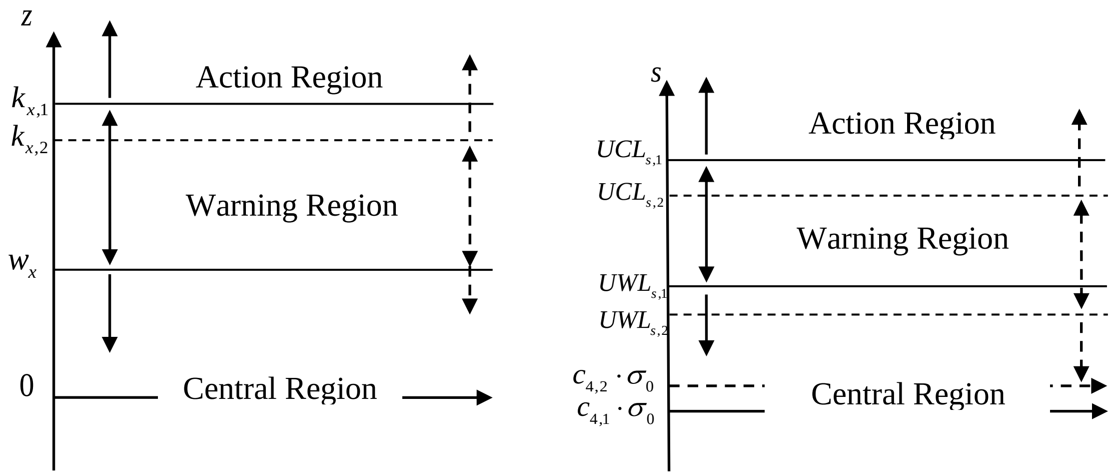


Figure 8-1: Regions of the VP_4 control scheme

The model state is fully defined by the actual state of the process, whose possible values are $Y_t = (0,0),...,(m,r)$, plus the failure state *F*, combined with the decision made at each transition step a_t ($a_t = 0$ if the statistic lies within the central zone, $a_t = 1$, in case of a warning and $a_t = 2$, in case of an *OOC* signal). Therefore, the state space includes $\left[(m+1)\times(r+1)\times 3+1\right]$ possible states. The transition probability matrix of the Markov chain is shown in Figure 8-2.

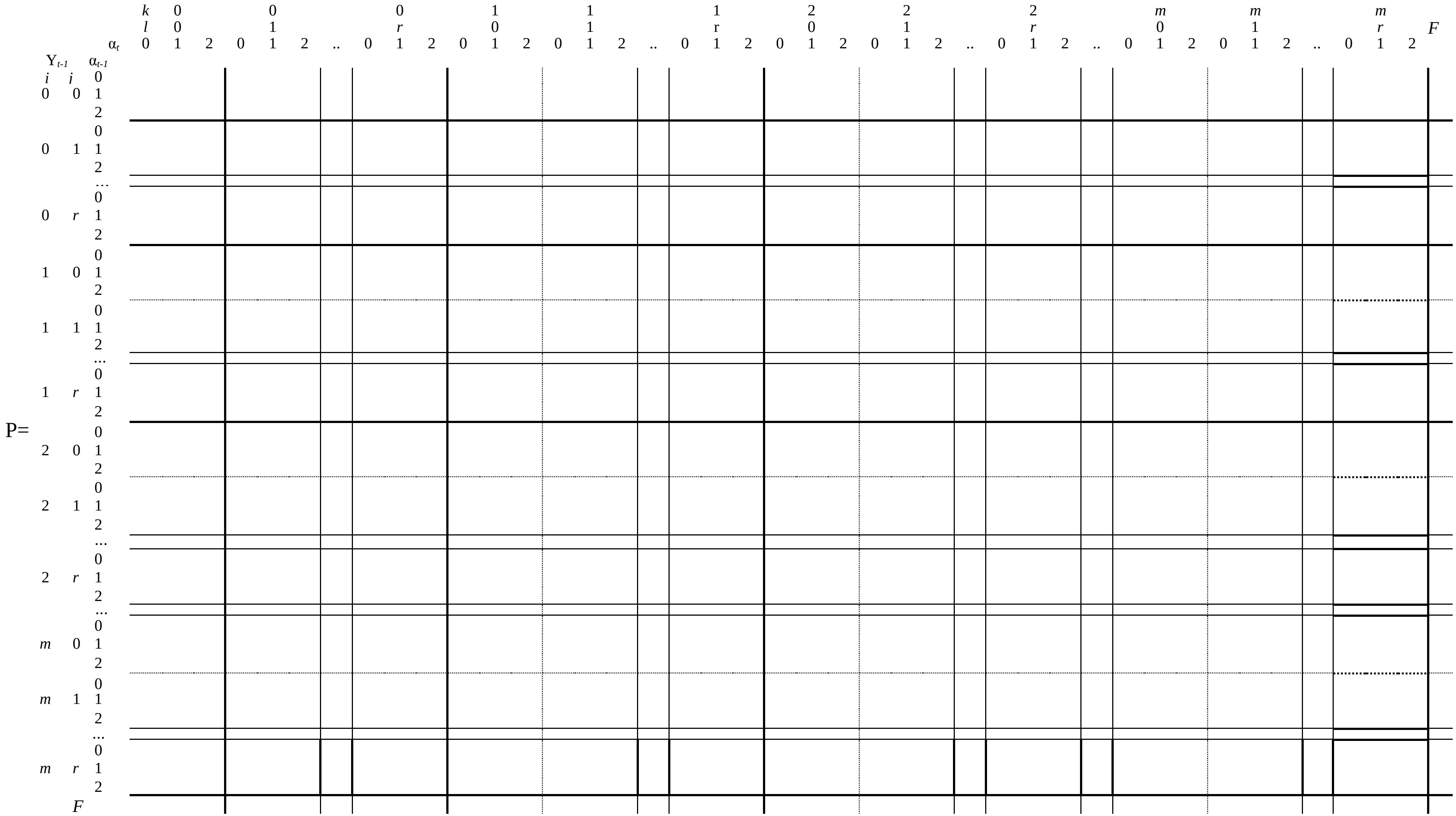


Figure 8-2: Transition Probability Matrix of the VP_4 control scheme

The computation of the probability of a process transition from state (i,j) to (k,l) $\forall i \neq k$ and $\forall j \neq l$ within an interval of duration h_q is based on equation (6.6), which is modified to account for the probability of a failure-free interval, and is derived from the following expression:

$$\begin{aligned} p_{\substack{(i,j)\\(k,l)}}(h_q) = & \int_0^{h_q} \sum_{y=i+1}^{k} \lambda_{x(i\to y)} \cdot \exp\left(-\left(v_{x,i} + v_{s,j} + \lambda_{Fx|i} + \lambda_{Fs|j}\right)\cdot t\right)\cdot p_{\substack{(y,j)\\(k,l)}}(h_q - t)dt + \\ & + \int_0^{h_q} \sum_{z=j+1}^{l} \lambda_{s(j\to z)} \cdot \exp\left(-\left(v_{x,i} + v_{s,j} + \lambda_{Fx|i} + \lambda_{Fs|j}\right)\cdot t\right)\cdot p_{\substack{(i,z)\\(k,l)}}(h_q - t)dt \end{aligned} \tag{8.1}$$

In the non-desirable case, where a failure occurs within a sampling interval of duration h_q, the process transits to state *F*, the production cycle ends, and afterwards, the process is restored to the *IC* state. A failure, when the process operates under (i,j), may either occur without the advent of any quality shift or after the deterioration of the process to an inferior state. The probability of a process transition to the failure state (state *F*), is:

$$\begin{aligned} p_{\substack{(i,j)\\F}}(h_q) = & \int_0^{h_q} \left(\lambda_{Fx|i} + \lambda_{Fs|j}\right)\cdot \exp\left(-\left(\lambda_{Fx|i} + \lambda_{Fs|j}\right)\cdot t\right)\cdot \exp\left(-\left(v_{x,i} + v_{s,j}\right)\cdot t\right)dt + \\ & + \int_0^{h_q} \sum_{y=i+1}^{m} \lambda_{x(i\to y)} \cdot \exp\left(-\left(v_{x,i} + v_{s,j} + \lambda_{Fx|i} + \lambda_{Fs|j}\right)\cdot t\right)\cdot p_{\substack{(y,j)\\F}}(h_q - t)dt + \\ & + \int_0^{h_q} \sum_{z=j+1}^{r} \lambda_{s(j\to z)} \cdot \exp\left(-\left(v_{x,i} + v_{s,j} + \lambda_{Fx|i} + \lambda_{Fs|j}\right)\cdot t\right)\cdot p_{\substack{(i,z)\\F}}(h_q - t)dt \end{aligned} \tag{8.2}$$

Additionally, the transition probabilities depend on the probability of each decision to be made at each transition step, which can easily be derived from the scheme's operation. Consequently, the exact expressions for the transition probabilities for a process moving from any state, i.e., (i,j), to any other state, i.e., (k,l), are (for $a_{t-1} = 0(1)$ q=1(2)):

$$
Prob_{\substack{(i,j)a_{t-1}\\(k,l)a_t}}\left(h_q\right)=\begin{cases}
p_{\substack{(i,j)\\(k,l)}}\left(h_q\right)\cdot\Phi\left(\dfrac{w_x-\delta_k\sqrt{n_q}}{\gamma_l}\right)\cdot P\left(\chi^2_{n_q\text{-}1}<\left(\dfrac{UWL_{s,q}}{\gamma_l\cdot\sigma_0}\right)^2\cdot\left(n_q-1\right)\right) & a_t=0\\
p_{\substack{(i,j)\\(k,l)}}\left(h_q\right)\cdot\left[\begin{array}{l}\Phi\left(\dfrac{k_{x,q}-\delta_k\sqrt{n_q}}{\gamma_l}\right)\cdot P\left(\chi^2_{n_q\text{-}1}<\left(\dfrac{UCL_{s,q}}{\gamma_l\cdot\sigma_0}\right)^2\cdot\left(n_q-1\right)\right)-\\ -\Phi\left(\dfrac{w_x-\delta_k\sqrt{n_q}}{\gamma_l}\right)\cdot P\left(\chi^2_{n_q\text{-}1}<\left(\dfrac{UWL_{s,q}}{\gamma_l\cdot\sigma_0}\right)^2\cdot\left(n_q-1\right)\right)\end{array}\right] & a_t=1\\
p_{\substack{(i,j)\\(k,l)}}\left(h_q\right)\cdot\left[1-\Phi\left(\dfrac{k_{x,q}-\delta_k\sqrt{n_q}}{\gamma_l}\right)\cdot P\left(\chi^2_{n_q\text{-}1}<\left(\dfrac{UCL_{s,q}}{\gamma_l\cdot\sigma_0}\right)^2\cdot\left(n_q-1\right)\right)\right] & a_t=2
\end{cases}
$$

(8.3)

In the special case of a process transition from any state (i,j) to failure state (state *F*): $Prob_{\substack{(i,j)a_{t-1}\\F}}\left(h_q\right)=p_{\substack{(i,j)\\F}}\left(h_q\right)$. On the other hand, whenever a failure occurs or an alarm is signaled the process resumes its operation in the *IC* state and the transition probabilities are $Prob_{\substack{F\\(k,l)a_t}}\left(h_q\right)=Prob_{\substack{(0,0)0\\(k,l)a_t}}\left(h_1\right)$.

Finally, in case an alarm is signaled $\left(a_{t-1}=2\right)$ and the process was actually operating *OOC* $\left(Y_t\neq(0,0)\right)$ the transition probabilities are derived from the following expression:

$$Prob_{\substack{(i,j)2\\(k,l)a_t}}\left(h_q\right)=Prob_{\substack{(0,0)0\\(k,l)a_t}}\left(h_1\right) \tag{8.4}$$

The limiting probability for the process transition to each possible state $Y_t=(k,l)\cup F\quad(k,l)=(0,0),...,(m,r)$, denoted by $\pi_{Y_ta_t}$ (in case of failure $\left(Y_t=F\right)$ $\pi_{Y_ta_t}$ is denoted by π_F) can be easily computed by solving the following linear system:

$$\pi_{Y_ta_t}=\sum_{Y_{t-1}=(0,0)\cup F}^{(m,r)\cup F}\sum_{a_{t-1}=0}^{2}Prob_{\substack{Y_{t-1}a_{t-1}\\Y_ta_t}}\left(h_q\right)\cdot\pi_{Y_{t-1}a_{t-1}}\text{ and }\sum_{Y_t=(0,0)\cup F}^{(m,r)\cup F}\sum_{a_t=0}^{2}\pi_{Y_ta_t}=1 \tag{8.5}$$

8.3 The Economic-Statistical Design

The mean cost of *OOC* operation for the proposed model is computed through a modification of equation (6.15) for processes where failures are possible to occur and is derived from the following expression:

$$\begin{aligned}
K_{(i,j)}(h_q) = {} & M_{(i,j)} \cdot h_q \cdot \exp\left(-\left(v_{x,i} + v_{s,j} + \lambda_{Fx|i} + \lambda_{Fs|j}\right)\cdot h_q\right) + \\
& + \int_0^{h_q} t \cdot M_{(i,j)} \cdot \left(\lambda_{Fx|i} + \lambda_{Fs|j}\right) \cdot \exp\left(-\left(\lambda_{Fx|i} + \lambda_{Fs|j}\right)\cdot t\right) \cdot \exp\left(-\left(v_{x,i} + v_{s,j}\right)\cdot t\right) dt + \\
& + \int_0^{h_q} \sum_{y=i+1}^{m} \left[\lambda_{x(i \to y)} \cdot \exp\left(-\left(v_{x,i} + v_{s,j} + \lambda_{Fx|i} + \lambda_{Fs|j}\right)\cdot t\right) \cdot \left(t \cdot M_{(i,j)} + K_{(y,j)}(h_q - t)\right)\right] dt + \\
& + \int_0^{h_q} \sum_{z=j+1}^{r} \left[\lambda_{s(j \to z)} \cdot \exp\left(-\left(v_{x,i} + v_{s,j} + \lambda_{Fx|i} + \lambda_{Fs|j}\right)\cdot t\right) \cdot \left(t \cdot M_{(i,j)} + K_{(i,z)}(h_q - t)\right)\right] dt
\end{aligned} \tag{8.6}$$

Consequently, the expected cycle cost can be computed from the following expression:

$$\begin{aligned}
EC = b & + \sum_{k=0}^{m}\sum_{l=0}^{r} \pi_{(k,l)0} \cdot \left(c \cdot n_1 + K_{(k,l)}\left(h_1\right)\right) + \sum_{k=0}^{m}\sum_{l=0}^{r} \pi_{(k,l)1} \cdot \left(c \cdot n_2 + K_{(k,l)}\left(h_2\right)\right) + \\
& + \sum_{k=0}^{m}\sum_{l=0}^{r} \pi_{(k,l)2} \cdot \left(c \cdot n_1 + K_{(0,0)}\left(h_1\right) + L_{(k,l)}\right) + \pi_F \cdot \left(c \cdot n_1 + K_{(0,0)}\left(h_1\right) + L_F\right)
\end{aligned} \tag{8.7}$$

The additional components of the expected cycle time length for the proposed model, are the following:

(a) The expected production time in the *IC* state $\tau_{IC/(0,0)}\left(h_q\right)$;

(b) The expected production time in the *OOC* state, when the process operates under the effect of state $\left(i, j\right)$ $i, j \geq 0$ at the beginning of the interval $\tau_{OOC/(i,j)}\left(h_q\right)$;

(c) The restoration time from failure T_F.

Specifically, the process operates in the *IC* state until either an assignable cause or a failure occurs. So, the expected *IC* time of the process within an interval of duration h_q, is derived from the following expression:

$$\tau_{IC/(0,0)}\left(h_q\right)=\int_0^{h_q}\exp\left(-\left(v_{x,0}+v_{s,0}+\lambda_{Fx|0}+\lambda_{Fs|0}\right)\cdot t\right)dt \tag{8.8}$$

Furthermore, in order to compute the *OOC* time when the process operates under the effect of state $\left(i,j\right)\neq\left(0,0\right)$ at the beginning of an interval of duration h_q, three alternative scenarios should be considered:

(a) The process remains under the effect of $\left(i,j\right)$ until a failure occurs;

(b) The process mean is further deteriorated from *i* to $y=i+1,...,m$;

(c) The standard deviation of the process is deteriorated from *j* to $z=j+1,...,r$.

In cases (b) and (c), the process operates under the effect of initial state $\left(i,j\right)$ for *t* time units and continues its *OOC* operation under an inferior state being the effect of the assignable cause that occurred.

Consequently, $\tau_{OOC/(i,j)}\left(h_q\right)$ can be derived from the following expression:

$$\begin{aligned}\tau_{OOC/(i,j)}\left(h_q\right)=&h_q\cdot\exp\left(-\left(v_{x,i}+v_{s,j}+\lambda_{Fx|i}+\lambda_{Fs|j}\right)\cdot h_q\right)+\\&+\int_0^{h_q}t\cdot(\lambda_{Fx|i}+\lambda_{Fs|j})\cdot\exp\left(-\left(v_{x,i}+v_{s,j}+\lambda_{Fx|i}+\lambda_{Fs|j}\right)\cdot t\right)dt+\\&+\int_0^{h_q}\sum_{y=i+1}^{m}\left[\lambda_{x(i\to y)}\cdot\exp\left(-\left(v_{x,i}+v_{s,j}+\lambda_{Fx|i}+\lambda_{Fs|j}\right)\cdot t\right)\cdot\left(t+\tau_{OOC/(y,j)}\left(h_q-t\right)\right)\right]dt+\\&+\int_0^{h_q}\sum_{z=j+1}^{r}\left[\lambda_{s(j\to z)}\cdot\exp\left(-\left(v_{x,i}+v_{s,j}+\lambda_{Fx|i}+\lambda_{Fs|j}\right)\cdot t\right)\cdot\left(t+\tau_{OOC/(i,z)}\left(h_q-t\right)\right)\right]dt\end{aligned} \tag{8.9}$$

Obviously, the *OOC* time when the process starts *IC* is simplified to the following equation:

$$\begin{aligned}\tau_{OOC/(0,0)}\left(h_q\right)=&\int_0^{h_q}\sum_{y=1}^{m}\left[\lambda_{x(0\to y)}\cdot\exp\left(-\left(v_{x,0}+v_{s,0}+\lambda_{Fx|0}+\lambda_{Fs|0}\right)\cdot t\right)\cdot\tau_{OOC/(y,0)}\left(h_q-t\right)\right]dt+\\&+\int_0^{h_q}\sum_{z=1}^{r}\left[\lambda_{s(0\to z)}\cdot\exp\left(-\left(v_{x,0}+v_{s,0}+\lambda_{Fx|0}+\lambda_{Fs|0}\right)\cdot t\right)\cdot\tau_{OOC/(0,z)}\left(h_q-t\right)\right]dt\end{aligned} \tag{8.10}$$

The expected duration of a transition step *ET* equals the sum of the *IC* time, in case: the process starts *IC*; an alarm is issued; a failure has occurred, plus the *OOC* production time, plus the time delay in case of a false alarm, plus the restoration times from an *OOC* state and a failure, all these multiplied by the long-term probability for each decision.

$$
\begin{aligned}
ET = {} & \pi_{(0,0)0} \cdot \tau_{IC/(0,0)}\left(h_1\right) + \pi_{(0,0)1} \cdot \tau_{IC/(0,0)}\left(h_2\right) + \sum_{k=0}^{m}\sum_{l=0}^{r} \pi_{(k,l)0} \cdot \tau_{OOC/(k,l)}\left(h_1\right) + \\
& + \sum_{k=0}^{m}\sum_{l=0}^{r} \pi_{(k,l)1} \cdot \tau_{OOC/(k,l)}\left(h_2\right) + \sum_{k=0}^{m}\sum_{l=0}^{r} \pi_{(k,l)2} \cdot \left(\tau_{IC/(0,0)}\left(h_1\right) + \tau_{OOC/(0,0)}\left(h_1\right) + T_{(k,l)}\right) + \\
& + \pi_F \cdot \left(\tau_{IC/(0,0)}\left(h_1\right) + \tau_{OOC/(0,0)}\left(h_1\right) + T_F\right)
\end{aligned}
\tag{8.11}
$$

8.4 Expected Availability

An important parameter of a production process's performance is the long-run expected availability, denoted by *EA*, which expresses the proportion of time the process is operable. From the renewal theory, *EA* may be expressed as the ratio of the expected time the equipment is available, denoted by *AT*, over the expected cycle time length, denoted by *ET* (*EA*=*AT*/*ET*).

The expected cycle time length *ET* has already been computed in Section 8.3, from equation (8.11).

As regards the expected time the equipment is available (*AT*), it can be computed as the sum of the expected *IC* time plus the *OOC* production time and can be derived from the following expression:

$$
\begin{aligned}
AT = {} & \pi_{(0,0)0} \cdot \tau_{IC/(0,0)}\left(h_1\right) + \pi_{(0,0)1} \cdot \tau_{IC/(0,0)}\left(h_2\right) + \sum_{k=0}^{m}\sum_{l=0}^{r} \pi_{(k,l)0} \cdot \tau_{OOC/(k,l)}\left(h_1\right) + \\
& + \sum_{k=0}^{m}\sum_{l=0}^{r} \pi_{(k,l)1} \cdot \tau_{OOC/(k,l)}\left(h_2\right) + \sum_{k=0}^{m}\sum_{l=0}^{r} \pi_{(k,l)2} \cdot \left(\tau_{IC/(0,0)}\left(h_1\right) + \tau_{OOC/(0,0)}\left(h_1\right)\right) + \\
& + \pi_F \cdot \left(\tau_{IC/(0,0)}\left(h_1\right) + \tau_{OOC/(0,0)}\left(h_1\right)\right)
\end{aligned}
\tag{8.12}
$$

8.5 Imperfect Process Restoration

A general framework of the extension of the proposed models to cases where the process restoration is imperfect has been developed in Section 3.4.2. In this section, the modification of the VP_4 control scheme under this assumption is presented in detail.

As it was explicitly stated in Chapter 3, after a confirmed quality shift upon inspection, the process may be restored to any non-inferior state with known (estimated) probabilities. Therefore, an imperfect *PM* action is considered. It should be noted that imperfect maintenance is usual in practical applications and received significant attention by scientists (Brown and Proschan, 1983, Pham and Wang, 1996, Ben-Daya, 1999, and Ben-Daya and Rahim, 2000).

Consequently, each time an alarm is signaled $\left(a_{t-1}=2\right)$ and the process was actually operating *OOC* $\left(Y_t \neq (0,0)\right)$ the transition probabilities equal the product of multiplying all the non-negative restoration probabilities to the respective transition probabilities after *PM* is performed.

$$Prob_{\substack{(i,j)2\\(k,l)a_t}}\left(h_q\right)=\sum_{i'=0}^{i}\sum_{j'=0}^{j} q_{\substack{(i,j)\\(i',j')}}\cdot Prob_{\substack{(i',j')0\\(k,l)a_t}}\left(h_q\right) \tag{8.13}$$

It should be noted that the model can be easily modified to consider tightened control as a protection against imperfect process restoration, especially if the restoration probabilities to *OOC* states are high enough to justify the extra cost imposed by considering a large size after the shortest interval for the first sample after a *PM* action $\left(Prob_{\substack{(i,j)2\\(k,l)a_t}}\left(h_q\right)=\sum_{i'=0}^{i}\sum_{j'=0}^{j} q_{\substack{(i,j)\\(i',j')}}\cdot Prob_{\substack{(i',j')0\\(k,l)a_t}}\left(h_2\right)\right)$.

Furthermore, both the expected cost and the expected duration of a transition step should be modified in case of imperfect process restoration and are, now, computed from the following expressions:

$$
\begin{aligned}
EC = b &+ \sum_{k=0}^{m}\sum_{l=0}^{r}\pi_{(k,l)0}\cdot\left(c\cdot n_1 + K_{(k,l)}\left(h_1\right)\right) + \sum_{k=0}^{m}\sum_{l=0}^{r}\pi_{(k,l)1}\cdot\left(c\cdot n_2 + K_{(k,l)}\left(h_2\right)\right) + \\
&+ \sum_{k=0}^{m}\sum_{l=0}^{r}\sum_{u=0}^{k}\sum_{v=0}^{l}\pi_{(k,l)2}\cdot q_{\substack{(k,l)\\(u,v)}}\cdot\left(c\cdot n_2 + K_{(u,v)}\left(h_2\right)\right) + \sum_{k=0}^{m}\sum_{l=0}^{r}\pi_{(k,l)2}\cdot L_{(k,l)} + \\
&+ \pi_F\cdot\left(c\cdot n_1 + K_{(0,0)}\left(h_1\right) + L_F\right)
\end{aligned} \tag{8.14}
$$

$$
\begin{aligned}
ET = {} & \pi_{(0,0)0}\cdot\tau_{IC/(0,0)}\left(h_1\right) + \pi_{(0,0)1}\cdot\tau_{IC/(0,0)}\left(h_2\right) + \sum_{k=0}^{m}\sum_{l=0}^{r}\pi_{(k,l)0}\cdot\tau_{OOC/(k,l)}\left(h_1\right) + \\
&+ \sum_{k=0}^{m}\sum_{l=0}^{r}\pi_{(k,l)1}\cdot\tau_{OOC/(k,l)}\left(h_2\right) + \sum_{k=0}^{m}\sum_{l=0}^{r}\pi_{(k,l)2}\cdot q_{\substack{(k,l)\\(0,0)}}\cdot\tau_{IC/(0,0)}\left(h_2\right) + \\
&+ \sum_{k=0}^{m}\sum_{l=0}^{r}\sum_{u=0}^{k}\sum_{v=0}^{l}\pi_{(k,l)2}\cdot q_{\substack{(k,l)\\(u,v)}}\cdot\tau_{OOC/(u,v)}\left(h_2\right) + \pi_F\cdot\left(\tau_{IC/(0,0)}\left(h_1\right) + \tau_{OOC/(0,0)}\left(h_1\right)\right) + \\
&+ \sum_{k=0}^{m}\sum_{l=0}^{r}\pi_{(k,l)2}\cdot T_{(k,l)} + \pi_F\cdot T_F
\end{aligned} \tag{8.15}
$$

It is obvious that the expected time the equipment is available should be also modified and is derived by the following equation:

$$
\begin{aligned}
AT = {} & \pi_{(0,0)0}\cdot\tau_{IC/(0,0)}\left(h_1\right) + \pi_{(0,0)1}\cdot\tau_{IC/(0,0)}\left(h_2\right) + \sum_{k=0}^{m}\sum_{l=0}^{r}\pi_{(k,l)0}\cdot\tau_{OOC/(k,l)}\left(h_1\right) \\
&+ \sum_{k=0}^{m}\sum_{l=0}^{r}\pi_{(k,l)1}\cdot\tau_{OOC/(k,l)}\left(h_2\right) + \sum_{k=0}^{m}\sum_{l=0}^{r}\pi_{(k,l)2}\cdot q_{\substack{(k,l)\\(0,0)}}\cdot\tau_{IC/(0,0)}\left(h_2\right) + \\
&+ \sum_{k=0}^{m}\sum_{l=0}^{r}\sum_{u=0}^{k}\sum_{v=0}^{l}\pi_{(k,l)2}\cdot q_{\substack{(k,l)\\(u,v)}}\cdot\tau_{OOC/(u,v)}\left(h_2\right) + \pi_F\cdot\left(\tau_{IC/(0,0)}\left(h_1\right) + \tau_{OOC/(0,0)}\left(h_1\right)\right)
\end{aligned} \tag{8.16}
$$

8.6 Optimization Problem

In this model we consider a multi-objective optimization problem where the goal is to find the optimal design parameters in order for two objective functions, i.e., minimize *ECT* and maximize *EA*, to be optimized subject to some statistical constraints. The general optimization problem is formulated as follows:

$$
\begin{gathered}
\min_{DP_q} ECT, \max_{DP_q} EA \\
\text{s.t. } h_1, h_2, n_1, n_2, w_x, k_{x,1}, k_{x,2}, w_s, k_{s,1}, k_{s,2} > 0 \\
h_2 \le h_1 \\
n_2 \ge n_1 \\
w_x \le k_{x,2} \le k_{x,1}
\end{gathered} \tag{8.17}
$$

$$w_s \leq k_{s,2} \leq k_{s,1}$$

$$n_1, n_2 \in \mathbb{Z}^+$$

$$\alpha \leq 0.02$$

In order to solve this bi-objective optimization problem we utilize the ε-constraint method introduced in Haimes *et al*. (1971), where one objective is selected to be optimized and the rest objectives are restricted within specified bounds.

Without loss of generality, we consider the minimization of *ECT* as the primary objective and set $EA \geq \varepsilon_2$ as an additional constraint to the optimization problem. The optimization problem is then reformulated as:

$$\begin{gathered}
\min_{DP_q} ECT \\
\text{s.t. } EA \geq \varepsilon_2 \\
h_1, h_2, n_1, n_2, w_x, k_{x,1}, k_{x,2}, w_s, k_{s,1}, k_{s,2} > 0 \\
h_2 \leq h_1 \\
n_2 \geq n_1 \\
w_x \leq k_{x,2} \leq k_{x,1} \\
w_s \leq k_{s,2} \leq k_{s,1} \\
n_1, n_2 \in \mathbb{Z}^+ \\
\alpha \leq 0.02
\end{gathered} \tag{8.18}$$

Through iterative increase of the value of ε_2 by a pre-defined constant *Δ*, weakly Pareto optima can be defined as feasible solutions of the optimization problem (equation (8.18)). The starting value of ε_2 should be selected by determining the optimal DP_q (DP_q^*) that minimizes *ECT*, without any constraints for *EA*, and then compute the excepted availability for this set of design parameters $\left(\varepsilon_2 = EA(DP_q^*)\right)$. On the other hand by optimizing only *EA*, the "best" allowable value of ε_2 can be defined. In case there is a unique solution of the optimization problem in the aforementioned range of ε_2 values, then, based on the research of Miettinen (1999), the solution is Pareto optimal.

It should be noted that the maximization of *EA* could, equivalently, be selected as the primary objective, especially for processes where the estimation of cost parameters is difficult, for example for relatively new processes and/or when availability plays a more significant role than cost, for example in military processes.

The minimization of *ECT* is achieved by means of a computer program developed in Fortran Power Station 4.0.

8.7 Numerical Analysis

In this section, the aforementioned approach for computing the optimum design parameters of the control scheme, the minimum expected cost per time unit and the maximum expected availability is applied to 64 cases for processes where two assignable causes that affect the process mean and that affect the standard deviation are possible to occur $\left(m=r=2\right)$. Each case is defined by thirteen process $\left(\lambda,\lambda_F,\delta,\gamma\right)$, economic $\left(c,b,M,L_{(0,0)},L,L_F\right)$ and time parameters $\left(T_{(0,0)},T,T_F\right)$ that vary at two levels.

It should be mentioned that $\lambda_{x(g\to u)}=\lambda_{s(g\to u)}=\lambda/2^{\left[(u-g)-1\right]}$, $\lambda_{Fx|g}=\lambda_{Fs|g}=\lambda/2^{(m-g)}$, where $0\le g\le 2$, $g<u\le 2$. Moreover, $\delta_1=\delta$, $\delta_2=1{,}5\cdot\delta$ and $\gamma_1=\gamma$, $\gamma_2=2\cdot\gamma-1$. Furthermore, $M_{(0,1)}=M_{(1,0)}=M$, $M_{(0,2)}=M_{(2,0)}=1.5M$ and $M_{(i,j)}=0.75\cdot\left(M_{(i,0)}+M_{(j,0)}\right)$ if $i,j\ge 1$.

In order to limit the large number of possible scenarios, i.e. $2^{13}=8192$ and for the sake of brevity, the variable sampling cost *c*, the cost of a *PM* and *CM* action $\left(L,L_F\right)$ and the time parameters $\left(T_{(0,0)},T,T_F\right)$ are assumed to remain constant. Specifically, *c*=10, $L_{(0,1)}=L_{(1,0)}=400$, $L_{(0,2)}=L_{(2,0)}=450$, $L_{(1,1)}=600$, $L_{(1,2)}=L_{(2,1)}=637.5$, $L_{(2,2)}=675$, $L_F=700$, $T_{(0,0)}=0.1$, $T_{(0,1)}=T_{(1,0)}=0.25$, $T_{(0,2)}=T_{(2,0)}=T_{(1,1)}=0.5$, $T_{(1,2)}=T_{(2,1)}=0.75$, $T_{(2,2)}=1.0$ and $T_F=1.25$. The benchmark of the process scenarios is presented in Table 8-1.

Table 8-1: Parameter sets of the 64 numerical examples for the VP_4 control scheme

Case	b	M	$L_{(0,0)}$	λ	δ	γ	Case	b	M	$L_{(0,0)}$	λ	δ	γ
1	0	100	100	0.01	0.5	1.414	33	0	100	100	0.01	0.5	2.0
2	0	100	200	0.01	0.5	1.414	34	0	100	200	0.01	0.5	2.0
3	0	1000	100	0.01	0.5	1.414	35	0	1000	100	0.01	0.5	2.0
4	0	1000	200	0.01	0.5	1.414	36	0	1000	200	0.01	0.5	2.0
5	5	100	100	0.01	0.5	1.414	37	5	100	100	0.01	0.5	2.0
6	5	100	200	0.01	0.5	1.414	38	5	100	200	0.01	0.5	2.0
7	5	1000	100	0.01	0.5	1.414	39	5	1000	100	0.01	0.5	2.0
8	5	1000	200	0.01	0.5	1.414	40	5	1000	200	0.01	0.5	2.0
9	0	100	100	0.1	0.5	1.414	41	0	100	100	0.1	0.5	2.0
10	0	100	200	0.1	0.5	1.414	42	0	100	200	0.1	0.5	2.0
11	0	1000	100	0.1	0.5	1.414	43	0	1000	100	0.1	0.5	2.0
12	0	1000	200	0.1	0.5	1.414	44	0	1000	200	0.1	0.5	2.0
13	5	100	100	0.1	0.5	1.414	45	5	100	100	0.1	0.5	2.0
14	5	100	200	0.1	0.5	1.414	46	5	100	200	0.1	0.5	2.0
15	5	1000	100	0.1	0.5	1.414	47	5	1000	100	0.1	0.5	2.0
16	5	1000	200	0.1	0.5	1.414	48	5	1000	200	0.1	0.5	2.0
17	0	100	100	0.01	1.5	1.414	49	0	100	100	0.01	1.5	2.0
18	0	100	200	0.01	1.5	1.414	50	0	100	200	0.01	1.5	2.0
19	0	1000	100	0.01	1.5	1.414	51	0	1000	100	0.01	1.5	2.0
20	0	1000	200	0.01	1.5	1.414	52	0	1000	200	0.01	1.5	2.0
21	5	100	100	0.01	1.5	1.414	53	5	100	100	0.01	1.5	2.0
22	5	100	200	0.01	1.5	1.414	54	5	100	200	0.01	1.5	2.0
23	5	1000	100	0.01	1.5	1.414	55	5	1000	100	0.01	1.5	2.0
24	5	1000	200	0.01	1.5	1.414	56	5	1000	200	0.01	1.5	2.0
25	0	100	100	0.1	1.5	1.414	57	0	100	100	0.1	1.5	2.0
26	0	100	200	0.1	1.5	1.414	58	0	100	200	0.1	1.5	2.0
27	0	1000	100	0.1	1.5	1.414	59	0	1000	100	0.1	1.5	2.0
28	0	1000	200	0.1	1.5	1.414	60	0	1000	200	0.1	1.5	2.0
29	5	100	100	0.1	1.5	1.414	61	5	100	100	0.1	1.5	2.0
30	5	100	200	0.1	1.5	1.414	62	5	100	200	0.1	1.5	2.0
31	5	1000	100	0.1	1.5	1.414	63	5	1000	100	0.1	1.5	2.0
32	5	1000	200	0.1	1.5	1.414	64	5	1000	200	0.1	1.5	2.0

The restoration probabilities of the process from any state (first row) to any other state (first column) are presented in Table 8-2.

Table 8-2: Restoration probabilities for the VP_4 control scheme

	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)	(2,0)	(2,1)	(2,2)
(0,0)	1	0	0	0	0	0	0	0	0
(0,1)	1	0	0	0	0	0	0	0	0
(0,2)	0.3	0.7	0	0	0	0	0	0	0
(1,0)	1	0	0	0	0	0	0	0	0
(1,1)	0.2	0.4	0	0.4	0	0	0	0	0
(1,2)	0.05	0.1	0.3	0.25	0.3	0	0	0	0
(2,0)	0.7	0	0	0.3	0	0	0	0	0
(2,1)	0.05	0.25	0	0.1	0.3	0	0.3	0	0
(2,2)	0.02	0.04	0.15	0.04	0.1	0.25	0.15	0.25	0

The optimum design parameters, the expected quality control cost, the expected availability and the measures of statistical performance for each scenario are presented in Table 8-3.

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Table 8-3: Economic-Statistical design for numerical examples 1-32: optimal control policy, cost, availability and related statistical measures for the VP_4 control scheme

Optimum Design Parameters													*Statistical Measures*						
Case	h_1	h_2	n_1	n_2	w_x	$k_{x,1}$	$k_{x,2}$	w_s	$k_{s,1}$	$k_{s,2}$	ECT_{VP4}	*EA*	α	$1\text{-}\beta$	*ANOF*	ARL_0	*WARL*	*ATC*	*EATR*
1	1.4	1.4	2	11	1.3	2.5	1.6	2.0	3.6	2.0	28.393	0.993	0.0200	0.2558	0.0062	50.08	3.91	48.84	15.51
2	1.5	1.5	2	12	1.2	2.6	1.6	2.0	3.6	2.0	29.008	0.993	0.0200	0.2773	0.0058	50.02	3.61	48.83	15.50
3	0.5	0.2	2	10	1.2	2.6	1.6	1.8	3.7	1.9	81.163	0.992	0.0199	0.2380	0.0211	50.23	4.20	47.73	14.39
4	0.5	0.3	2	11	1.2	2.6	1.6	1.8	3.7	1.9	83.262	0.992	0.0199	0.2509	0.0203	50.38	3.99	47.94	14.61
5	1.7	1.7	2	13	1.2	2.6	1.7	1.8	3.7	1.9	30.183	0.993	0.0200	0.2981	0.0051	50.05	3.35	48.91	15.58
6	1.7	1.7	2	13	1.2	2.6	1.7	1.8	3.7	1.9	30.689	0.993	0.0200	0.2981	0.0051	50.05	3.35	48.91	15.58
7	0.5	0.4	2	11	1.2	2.6	1.6	1.8	3.7	1.9	86.663	0.992	0.0198	0.2541	0.0195	50.40	3.94	48.16	14.82
8	0.5	0.5	2	12	1.2	2.6	1.6	1.8	3.7	1.9	88.496	0.992	0.0198	0.2670	0.0187	50.53	3.75	48.34	15.00
9	0.4	0.4	2	58	2.1	2.2	2.2	3.1	3.3	3.1	115.314	0.941	0.0187	0.2735	0.0149	53.43	3.66	5.73	2.40
10	0.6	0.6	2	62	2.2	2.3	2.3	2.8	3.0	2.9	116.940	0.943	0.0190	0.3201	0.0091	52.77	3.12	5.98	2.65
11	0.2	0.2	2	14	1.1	2.7	1.7	1.7	3.7	2.0	264.800	0.939	0.0199	0.3282	0.0387	50.27	3.05	5.36	2.03
12	0.2	0.2	2	14	1.1	2.7	1.7	1.7	3.7	2.0	268.669	0.939	0.0199	0.3282	0.0387	50.27	3.05	5.36	2.03
13	0.5	0.5	2	70	2.2	2.3	2.3	3.0	3.0	3.0	118.837	0.943	0.0190	0.2939	0.0114	52.64	3.40	5.87	2.55
14	0.6	0.6	2	56	2.1	2.3	2.1	2.8	3.0	2.8	122.752	0.943	0.0200	0.3290	0.0096	50.08	3.04	5.93	2.60
15	0.2	0.2	2	13	1.1	2.7	1.7	1.7	3.7	2.0	277.308	0.939	0.0199	0.3184	0.0386	50.18	3.14	5.37	2.04
16	0.2	0.2	2	13	1.1	2.7	1.7	1.7	3.7	2.0	281.164	0.939	0.0199	0.3184	0.0386	50.18	3.14	5.37	2.04
17	1.4	1.4	2	6	1.7	2.5	2.2	1.6	3.2	1.7	21.473	0.993	0.0198	0.3509	0.0064	50.53	2.85	48.20	14.87
18	1.5	1.5	2	7	1.8	2.5	2.4	1.6	3.2	1.6	22.037	0.993	0.0200	0.3626	0.0060	50.08	2.76	48.20	14.87
19	0.5	0.2	2	6	1.7	2.6	2.3	1.4	3.3	1.5	61.090	0.992	0.0200	0.3312	0.0211	50.10	3.02	47.61	14.28
20	0.5	0.3	2	6	1.7	2.6	2.3	1.4	3.3	1.5	63.168	0.992	0.0200	0.3343	0.0204	50.11	2.99	47.81	14.47
21	1.7	1.7	2	8	1.7	2.5	2.3	1.6	3.3	1.6	23.168	0.993	0.0200	0.3824	0.0053	50.05	2.61	48.20	14.87
22	1.8	1.8	2	8	1.8	2.5	2.4	1.5	3.3	1.6	23.686	0.993	0.0199	0.3864	0.0049	50.20	2.59	48.21	14.87
23	0.6	0.3	2	7	1.6	2.5	2.4	1.3	3.4	1.6	66.453	0.992	0.0200	0.3576	0.0175	50.03	2.80	47.69	14.36
24	0.6	0.4	2	7	1.6	2.5	2.4	1.3	3.4	1.6	68.186	0.992	0.0200	0.3608	0.0169	50.05	2.77	47.85	14.51
25	0.4	0.4	2	9	1.8	2.5	2.4	1.6	3.4	1.7	92.639	0.939	0.0193	0.4362	0.0175	51.69	2.29	5.35	2.01
26	0.4	0.4	2	12	2.2	2.4	2.3	1.9	3.1	1.9	93.522	0.939	0.0198	0.4206	0.0178	50.46	2.38	5.35	2.01
27	0.2	0.2	2	8	1.6	2.5	2.4	1.3	3.4	1.7	201.088	0.939	0.0198	0.4118	0.0401	50.43	2.43	5.27	1.94
28	0.2	0.2	2	9	1.8	2.5	2.4	1.4	3.4	1.6	204.988	0.939	0.0200	0.4082	0.0403	50.09	2.45	5.27	1.94
29	0.5	0.4	2	12	2.0	2.5	2.4	1.7	3.3	1.7	98.784	0.939	0.0199	0.4558	0.0145	50.13	2.19	5.37	2.04
30	0.6	0.6	2	25	2.2	2.3	2.3	2.0	3.3	2.0	99.446	0.939	0.0196	0.5033	0.0110	51.14	1.99	5.40	2.06
31	0.2	0.2	2	9	1.8	2.5	2.5	1.4	3.4	1.6	213.382	0.939	0.0200	0.4082	0.0403	50.09	2.45	5.27	1.94
32	0.2	0.2	2	8	1.6	2.5	2.4	1.3	3.4	1.7	217.515	0.939	0.0198	0.4118	0.0401	50.43	2.43	5.27	1.94

Table 8-4: Economic-Statistical design for numerical examples 33-64: optimal control policy, cost, availability and related statistical measures for the VP_4 control scheme

	Optimum Design Parameters												Statistical Measures							
Case	h_1	h_2	n_1	n_2	w_x	$k_{x,1}$	$k_{x,2}$	w_s	$k_{s,1}$	$k_{s,2}$	ECT_{VP4}	*EA*	α	$1-\beta$	*ANOF*	ARL_0	*WARL*	*ATC*	*EATR*
33	1.5	1.5	2	8	1.0	2.5	1.5	2.6	3.9	2.7	23.342	0.993	0.0200	0.3536	0.0059	50.07	2.83	48.74	15.40
34	1.6	1.6	2	9	1.1	2.6	1.4	2.5	3.9	2.7	23.869	0.993	0.0199	0.3600	0.0055	50.20	2.78	48.71	15.37
35	0.5	0.3	2	8	1.1	2.6	1.3	2.6	3.9	2.8	66.357	0.992	0.0199	0.3163	0.0203	50.32	3.16	47.91	14.58
36	0.6	0.2	2	8	1.0	2.6	1.4	2.3	4.0	2.8	68.352	0.992	0.0200	0.3278	0.0183	50.10	3.05	47.60	14.27
37	1.8	1.8	2	9	1.0	2.6	1.5	2.3	3.9	2.6	24.896	0.993	0.0199	0.3805	0.0049	50.14	2.63	48.81	15.48
38	1.9	1.9	2	10	1.0	2.6	1.5	2.4	3.8	2.7	25.373	0.993	0.0198	0.3947	0.0046	50.42	2.53	48.80	15.46
39	0.6	0.4	2	9	1.0	2.6	1.4	2.5	3.8	2.8	71.280	0.992	0.0200	0.3483	0.0169	50.03	2.87	47.99	14.65
40	0.6	0.4	2	10	1.0	2.6	1.4	2.5	3.8	2.8	72.958	0.992	0.0200	0.3663	0.0164	50.11	2.73	48.15	14.82
41	0.4	0.4	2	47	2.1	2.1	2.1	3.7	3.8	3.7	95.018	0.940	0.0197	0.3417	0.0167	50.84	2.93	5.54	2.20
42	0.4	0.4	2	68	2.1	2.1	2.1	3.7	3.8	3.7	96.230	0.940	0.0197	0.3472	0.0168	50.84	2.88	5.53	2.20
43	0.2	0.2	2	10	0.9	2.6	1.6	2.4	3.9	2.6	217.454	0.940	0.0189	0.4063	0.0378	52.89	2.46	5.34	2.00
44	0.2	0.2	2	10	1.0	2.6	1.5	2.4	3.8	2.7	219.965	0.940	0.0200	0.3968	0.0398	50.11	2.52	5.34	2.00
45	0.6	0.6	2	68	2.1	2.1	2.1	3.6	3.7	3.6	101.454	0.941	0.0200	0.4021	0.0104	50.01	2.49	5.67	2.34
46	0.6	0.6	2	66	2.1	2.1	2.1	3.6	3.7	3.6	102.511	0.941	0.0200	0.4018	0.0104	50.01	2.49	5.67	2.34
47	0.2	0.2	2	10	0.9	2.6	1.6	2.4	3.9	2.6	229.747	0.940	0.0189	0.4063	0.0378	52.89	2.46	5.34	2.00
48	0.2	0.2	2	9	0.9	2.6	1.6	2.2	3.7	2.6	232.723	0.940	0.0199	0.3968	0.0396	50.29	2.52	5.35	2.01
49	1.6	1.6	2	18	2.2	2.3	2.3	3.0	3.0	3.0	16.212	0.993	0.0195	0.5421	0.0056	47.92	1.84	47.92	14.58
50	1.8	1.8	2	11	2.1	2.2	2.1	2.5	3.3	2.5	16.821	0.993	0.0197	0.5668	0.0050	50.75	1.76	47.89	14.56
51	0.6	0.2	2	5	1.7	2.3	2.1	2.0	3.2	2.0	45.493	0.992	0.0198	0.5403	0.0171	50.58	1.85	47.51	14.18
52	0.6	0.1	2	5	1.7	2.7	2.3	1.9	3.7	2.2	46.567	0.993	0.0086	0.4864	0.0076	116.27	2.06	47.78	14.45
53	1.9	1.9	2	7	1.8	2.3	2.2	2.1	3.1	2.2	17.740	0.993	0.0199	0.5858	0.0048	50.21	1.71	47.95	14.62
54	2.0	2.0	2	7	1.8	2.3	2.2	2.1	3.1	2.2	18.197	0.993	0.0200	0.5875	0.0046	50.11	1.70	47.95	14.62
55	0.6	0.6	2	6	1.8	2.3	2.0	1.9	3.2	2.0	49.795	0.992	0.0200	0.5640	0.0161	50.01	1.77	48.06	14.73
56	0.7	0.1	2	6	1.7	2.7	2.4	1.9	3.7	2.2	50.970	0.993	0.0086	0.5043	0.0063	120.22	1.98	47.79	14.45
57	0.4	0.4	2	15	2.2	2.3	2.2	3.0	3.1	3.0	71.731	0.940	0.0197	0.5887	0.0190	50.66	1.70	5.25	1.92
58	0.4	0.4	2	16	2.1	2.2	2.1	3.1	3.3	3.1	74.034	0.940	0.0192	0.5947	0.0186	52.05	1.68	5.25	1.91
59	0.2	0.2	2	7	1.8	2.3	2.1	2.1	3.2	2.1	150.678	0.940	0.0197	0.5870	0.0414	50.72	1.70	5.23	1.89
60	0.2	0.2	2	7	1.7	2.3	2.2	2.1	3.2	2.1	154.880	0.940	0.0195	0.5931	0.0409	51.34	1.69	5.23	1.90
61	0.5	0.4	2	17	2.2	2.3	2.2	2.9	3.0	2.9	77.801	0.940	0.0194	0.6073	0.0149	51.49	1.65	5.27	1.93
62	0.6	0.6	2	17	2.2	2.3	2.2	2.9	3.0	2.9	79.053	0.940	0.0194	0.6314	0.0116	51.62	1.58	5.29	1.96
63	0.2	0.2	2	6	1.7	2.3	2.2	1.9	3.2	2.1	162.837	0.940	0.0198	0.5899	0.0416	50.48	1.70	5.23	1.90
64	0.2	0.2	2	7	1.7	2.3	2.2	2.1	3.2	2.1	166.986	0.940	0.0195	0.5931	0.0409	51.34	1.69	5.23	1.90

The examination of the results presented above leads to the following useful conclusions.

It is apparent that larger values of *OOC* operation costs *M* and/or larger values of occurrence rates λ, lead to larger values of *ECT*. Furthermore, larger values of *M* and/or λ, dictate smaller values of h_1 and h_2.

The logical explanation of the aforementioned conclusions is that in such cases, a more "expensive" policy of frequent sampling and/or "tighter" threshold and control limits is needed in order to avoid a long-lasting *OOC* operation

Another conclusion is that the effects of the assignable causes to the mean and/or the standard deviation of the process, δ and γ, respectively, are inversely correlated to the values of *ECT* and the values of *Type I* errors α. This conclusion can be explained intuitively. In case of lower shifts of the assignable causes to the process, the difficulty of the scheme to identify an alarm and to distinguish whether this alarm is a true or a false one increases, which, inevitably increases *ECT* and *Type I* error a.

Finally, the expected availability is lower in case of high occurrence rates, because of higher failure rates, which, unavoidably, increase the downtimes of the equipment and as a result lower the expected availability.

The comparison of the economic performance of the proposed VP_4 control scheme with less sophisticated control schemes, is given in Tables 8-5 and 8-6. Specifically, the VP_4 control scheme is compared to the respective: (a) *FP* control scheme; (b) *VSS* control scheme; (c) *VSI* control scheme; (d) *VSSI* control scheme.

Table 8-5: Economic comparison between the VP_4 control scheme and other less adaptive control schemes. Numerical examples 1-32

Case	*ECT* *FP*	*VSS*	*VSI*	*VSSI*	VP_4	$\frac{FP-VP_4}{FP}$ (%)	$\frac{VSS-VP_4}{VSS}$ (%)	$\frac{VSI-VP_4}{VSI}$ (%)	$\frac{VSSI-VP_4}{VSSI}$ (%)
1	33.592	31.654	31.394	29.501	28.393	15.48	10.30	9.56	3.76
2	34.647	32.627	32.388	30.966	29.008	16.28	11.09	10.44	6.32
3	99.189	96.501	90.508	87.551	81.163	18.17	15.89	10.33	7.30
4	103.876	99.426	93.947	90.644	83.262	19.84	16.26	11.37	8.14
5	37.002	34.779	34.696	34.006	30.183	18.43	13.21	13.01	11.24
6	37.924	35.617	35.541	35.095	30.689	19.08	13.84	13.65	12.55
7	109.725	105.017	100.181	96.144	86.663	21.02	17.48	13.49	9.86
8	112.177	107.942	103.243	99.237	88.496	21.11	18.02	14.28	10.82
9	135.062	122.747	115.314	115.314	115.314	14.62	6.06	0.00	0.00
10	135.259	124.475	124.475	117.265	116.940	13.54	6.05	6.05	0.28
11	311.407	301.820	292.677	291.422	264.800	14.97	12.27	9.52	9.14
12	318.809	304.589	299.591	298.910	268.669	15.73	11.79	10.32	10.12
13	135.610	131.534	118.837	118.837	118.837	12.37	9.65	0.00	0.00
14	135.839	132.710	132.710	124.014	122.752	9.63	7.50	7.50	1.02
15	337.457	311.860	315.047	298.027	277.308	17.82	11.08	11.98	6.95
16	344.859	314.603	324.486	301.968	281.164	18.47	10.63	13.35	6.89
17	23.013	22.817	22.533	22.533	21.473	6.69	5.89	4.70	4.70
18	23.780	23.561	23.297	23.250	22.037	7.33	6.47	5.41	5.22
19	67.100	66.933	64.547	64.318	61.090	8.96	8.73	5.36	5.02
20	69.533	69.304	67.046	66.718	63.168	9.15	8.85	5.78	5.32
21	25.286	25.036	24.805	24.562	23.168	8.38	7.46	6.60	5.68
22	25.960	25.703	25.482	25.151	23.686	8.76	7.85	7.05	5.82
23	73.571	73.330	71.163	70.599	66.453	9.68	9.38	6.62	5.87
24	75.928	75.678	73.511	72.272	68.186	10.20	9.90	7.24	5.65
25	96.982	95.241	95.003	95.003	92.639	4.48	2.73	2.49	2.49
26	99.334	97.202	97.202	97.018	93.522	5.85	3.79	3.79	3.60
27	211.016	209.510	209.510	209.510	201.088	4.70	4.02	4.02	4.02
28	219.199	217.635	217.635	216.857	204.988	6.48	5.81	5.81	5.47
29	104.741	102.445	102.445	102.445	98.784	5.69	3.57	3.57	3.57
30	106.394	104.062	104.062	104.062	99.446	6.53	4.44	4.44	4.44
31	235.438	233.759	230.834	226.459	213.382	9.37	8.72	7.56	5.77
32	243.621	239.860	236.100	230.815	217.515	10.72	9.32	7.87	5.76

Table 8-6: Economic comparison between the VP_4 control scheme and other less adaptive control schemes. Numerical examples 33-64

Case	*ECT* *FP*	*VSS*	*VSI*	*VSSI*	VP_4	$\frac{FP-VP_4}{FP}$ (%)	$\frac{VSS-VP_4}{VSS}$ (%)	$\frac{VSI-VP_4}{VSI}$ (%)	$\frac{VSSI-VP_4}{VSSI}$ (%)
33	29.015	24.636	26.623	24.284	23.342	19.55	5.25	12.32	3.88
34	29.832	25.214	27.469	25.025	23.869	19.99	5.33	13.11	4.62
35	85.936	71.331	74.365	70.964	66.357	22.78	6.97	10.77	6.49
36	88.241	72.981	77.256	72.884	68.352	22.54	6.34	11.53	6.22
37	31.215	26.286	28.607	26.147	24.896	20.24	5.29	12.97	4.78
38	31.782	26.763	29.347	26.591	25.373	20.17	5.19	13.54	4.58
39	92.362	76.167	82.160	75.905	71.280	22.83	6.42	13.24	6.09
40	94.667	77.724	84.890	77.575	72.958	22.93	6.13	14.06	5.95
41	114.815	98.031	104.231	95.706	95.018	17.24	3.07	8.84	0.72
42	116.934	99.433	106.259	96.591	96.230	17.71	3.22	9.44	0.37
43	264.410	231.576	238.775	229.589	217.454	17.76	6.10	8.93	5.29
44	272.172	234.681	246.931	234.648	219.965	19.18	6.27	10.92	6.26
45	123.103	103.773	113.010	102.201	101.454	17.59	2.23	10.23	0.73
46	124.449	104.472	115.038	102.641	102.511	17.63	1.88	10.89	0.13
47	289.290	243.528	262.841	241.886	229.747	20.58	5.66	12.59	5.02
48	293.035	247.044	270.457	247.010	232.723	20.58	5.80	13.95	5.78
49	17.495	16.212	17.102	16.212	16.212	7.33	0.00	5.20	0.00
50	18.090	16.953	17.713	16.953	16.821	7.01	0.78	5.04	0.78
51	48.880	45.770	47.101	45.770	45.493	6.93	0.61	3.41	0.61
52	50.788	47.650	48.960	47.201	46.567	8.31	2.27	4.89	1.34
53	19.236	18.040	18.840	18.040	17.740	7.78	1.66	5.84	1.66
54	19.778	18.203	19.393	18.203	18.197	7.99	0.03	6.17	0.03
55	53.942	50.395	52.076	50.295	49.795	7.69	1.19	4.38	0.99
56	55.850	51.952	53.878	51.561	50.970	8.74	1.89	5.40	1.15
57	79.205	72.566	76.480	72.566	71.731	9.44	1.15	6.21	1.15
58	81.956	74.139	78.437	74.139	74.034	9.67	0.14	5.61	0.14
59	167.227	151.570	158.660	151.005	150.678	9.90	0.59	5.03	0.22
60	171.251	156.763	163.606	156.763	154.880	9.56	1.20	5.33	1.20
61	86.447	78.719	83.106	77.869	77.801	10.00	1.17	6.38	0.09
62	87.870	80.446	84.895	80.081	79.053	10.03	1.73	6.88	1.28
63	179.375	165.430	173.149	165.430	162.837	9.22	1.57	5.96	1.57
64	183.399	167.050	177.899	167.050	166.986	8.95	0.04	6.13	0.04

The cost savings achieved by the implementation of the proposed control chart are on average 13.24% compared to the *FP* control chart; 6.18% compared to the *VSS* control chart; 8.10% compared to the *VSI* control chart and 4.09% compared to the VSSI control chart. The economic improvement is inversely related to λ and is greater in case M is large.

9. *VP* MULTIVARIATE CONTROL SCHEME FOR PROCESSES SUBJECT TO MULTIPLE QUALITY SHIFTS AFFECTING BOTH LOCATION AND SCALE (*VP*$_5$)

9.1 Introduction

In this chapter, we investigate the economic-statistical design of a *VP* control scheme for monitoring multivariate processes, when multiple assignable causes may shift the location and/or the variability of the correlated quality characteristics.

By considering m_{mv} possible assignable causes for the mean vector, there are $p \times m_{mv}$ possible values of $\delta_{i,\rho}$, for every quality characteristic ρ $\left(\rho = 1, 2, ..., p\right)$ being shifted from any possible assignable cause i $\left(i = 1, ..., m_{mv}\right)$, i.e., $\delta_{i,\rho} \in \left(\delta_{1,1}, ..., \delta_{1,p}; ...; \delta_{m_{mv},1}, ..., \delta_{m_{mv},p}\right)$. In a similar manner, for r_{cm} assignable causes as regards the covariance matrix, the $p \times r_{cm}$ possible values for the magnitude of a shift are $\gamma_{j,\rho} \in \left(\gamma_{1,1}, ..., \gamma_{1,p}; ...; \gamma_{r_{cm},1}, ..., \gamma_{r_{cm},p}\right)$. Apparently, in case the process operates in the *IC* state, then $\delta_{0,\rho} = 0$ and $\gamma_{0,\rho} = 1$, for every $\rho = 1, 2, ..., p$.

Specifically, a *VP* T^2 Hotelling's control chart is employed to monitor the mean vector through the statistic $T_q^2 = n_q \cdot \left(\bar{x}' - \mu_0\right)^{\mathrm{T}} \cdot \Sigma^{-1} \cdot \left(\bar{x}' - \mu_0\right)$, where $\bar{x}'$ is the sample mean vector derived from a collected sample of size n_q. Whenever the process operates *IC*, the T^2 statistic is a continuous, random variable that follows a chi-square distribution with *p* degrees of freedom. In case the process operates *OOC*, the T^2 statistic is distributed as a non-central chi-square distribution with *p* degrees of freedom and non-centrality parameter $n_q \cdot d_{(i,j)}^2$, where $d_{(i,j)} = \left(\bar{x}' - \mu_0'\right)^{\mathrm{T}} \cdot$ $\cdot \Sigma_j^{-1} \cdot \left(\bar{x}' - \mu_0'\right)$ represents the Mahalabonis distance of a sample mean vector $\bar{x}'$ from μ_0', when the mean vector and the covariance matrix are under the effect of assignable causes *i* and *j*, respectively $\left(\mu' = \mu_i', \Sigma = \Sigma_j\right)$.

Regarding the covariance matrix, a fully adaptive Shewhart control chart is employed to monitor the variability of a *p*-dimensional multivariate variable X_m, based on the value of its entropy. This chart is an extension of the entropy approach for the simple Shewhart control chart proposed by Guerrero-Cusumano (1995) to the adaptive case.

The statistic $E_{q,t} = \sqrt{\frac{2\cdot\left(n_q - 1\right)}{p}} \cdot \sum_{\rho=1}^{p} \ln\left(\frac{R_\rho}{\sigma_{0,\rho}}\right)$ is utilized in order to monitor the dispersion of the multivariate normal distribution and estimates the difference between the sample entropy and theoretical entropy. The range of the ρ^{th} quality characteristic, denoted by R_ρ, is derived from a collected sample of size n_q, i.e., $R_\rho = \max\left[x_{1,\rho}, x_{2,\rho}, ..., x_{n_q,\rho}\right] - \min\left[x_{1,\rho}, x_{2,\rho}, ..., x_{n_q,\rho}\right]$ and $\sigma_{0,\rho}$ is the in-control standard deviation of the ρ^{th} quality characteristic.

The E_q statistic is assumed to follow a normal distribution with mean μ_{0,E_q} and variance σ^2_{0,E_q} derived from the general expressions:

$$\mu_{0,E_q} = \sqrt{\frac{2\cdot(n_q - 1)}{p}} \cdot \sum_{\rho=1}^{p} E\left(\ln\left(\frac{R_\rho}{\sigma_{0,\rho}}\right)\right) \tag{9.1}$$

$$\sigma^2_{0,E_q} = \frac{2\cdot(n_q - 1)}{p} \cdot \left(\sum_{\rho=1}^{p} Var\left(\ln\left(\frac{R_\rho}{\sigma_{0,\rho}}\right)\right) + \sum_{\rho\neq f}^{p} Cov\left(\ln\left(\frac{R_\rho}{\sigma_{0,\rho}}\right), \ln\left(\frac{R_f}{\sigma_{0,f}}\right)\right)\right) \tag{9.2}$$

After some complicated mathematical manipulation, Guerrero-Cusumano (1995) concluded that μ_{0,E_q} and σ^2_{0,E_q} can be evaluated through the following equations:

$$\mu_{0,E_q} = \sqrt{2\cdot p\cdot(n_q - 1)} \cdot E\left(y\right) \tag{9.3}$$

$$\sigma^2_{0,E_q} = 2\cdot(n_q - 1)\cdot Var(y) + \frac{1}{p}\cdot\left[E(y)\right]^2 \cdot Tr\left(P_0 - I\right)^2 \tag{9.4}$$

where, $y = \ln\left(R_\rho / \sigma_{0,\rho}\right)$, Tr(A) is the trace of matrix A, I the identity matrix of size $p \times p$, $E\left(y\right) = \varepsilon\cdot\left[\ln\left(2/c\right) + \psi\left(\beta/2\right)\right]$ and $Var\left(y\right) = \varepsilon^2 \cdot \psi_1\left(\beta/2\right)$ (where ψ and

ψ_1 are the first (digamma function) and second (trigamma function) derivative of the natural algorithm of the gamma function, respectively, and ε, c, β constants).

The values of ε, c, β and, so, the values of $E(y)$ and $Var(y)$ can be found in Cadwell (1953), for any sample size n_q $(2 \le n_q \le 20)$ and have been tabulated below (Table 9-1). The range of the possible sample sizes $(n_q = 2, ..., 20)$ is consistent with the contemporary industry policies that dictate frequent sampling of relatively small sample sizes (Montgomery, 2009).

Table 9-1: Mean $(E(y))$ and variance $(Var(y))$ of variable $y = \ln(R_\rho / \sigma_{0,\rho})$

n_q	$E(y)$	$Var(y)$	n_q	$E(y)$	$Var(y)$
2	0.2886	1.2337	**12**	1.15204	0.0599668
3	0.3576	0.4123	**13**	1.17759	0.0556675
4	0.617684	0.237065	**14**	1.20026	0.0520683
5	0.768099	0.166459	**15**	1.22057	0.0490106
6	0.869731	0.129121	**16**	1.23921	0.0464001
7	0.944558	0.106262	**17**	1.25585	0.0440956
8	1.00293	0.0909723	**18**	1.27150	0.042082
9	1.05018	0.0797	**19**	1.28538	0.0402908
10	1.08952	0.0716104	**20**	1.29861	0.0387774
11	1.12295	0.0651794			

The assumption that E_q is a normally distributed variable is determined through a normality test. In specific, a simulation study for the distribution of E_q has been conducted with 10000 random samples for different values of sample size $(n_q = 2, ..., 20)$ and different correlation coefficients $(r_c = \pm 0.1, \pm 0.2, ..., \pm 0.9, \pm 0.95, \pm 0.99)$ when two $(p = 2)$ or three $(p = 3)$ quality characteristics are monitored simultaneously. The normality of E_q has been tested through the chi-square goodness of fit test. For the bivariate case $(p = 2)$, and for every possible combination of n_q and r_c the null hypothesis that E_q is normally distributed fails to be rejected at a 5% significance level. For the trivariate case

$(p=3)$, the normality assumption is appropriate for the statistic E_q for every n_q and r_c, except when high correlation is present $(r_c = \pm 0.95)$; in such cases a sample of size $n_q \geq 4$ is required.

It should be mentioned that for the chart that monitors the covariance matrix of the process, the standardized value of statistic E_q is utilized, denoted by z_{E_q}.

The remainder of this chapter is organized as follows. In Section 9.2, we present the mathematical model. Section 9.3 contains the design of the proposed control scheme. Section 9.4 presents the optimization problem and Section 9.5 provides a real example from the aircraft industry in order to illustrate the application of the proposed scheme.

It should be noted that this chapter uses material from Tasias and Nenes (2016c).

9.2 Mathematical Model

The proposed control scheme is denoted by VP_5, and as it is already mentioned in Chapter 3, is fully defined by the following design parameters $\{n_q, h_q, w_{mv,q}, k_{mv,q}, w_{cm,q}, k_{cm,q}\}$.

However, in real applications, the difficulty in computing the value of the mean or variance of each quality characteristic is not the same. For example, in a chemical/pharmaceutical industry, computing the mean weight of a predefined sample of pills is much easier, less time-consuming and with lower cost than computing the mean percentage portion of an ingredient of a pill. So, in order to achieve even better economic results, different sample sizes (subsamples) of each quality characteristic, within a collected sample of size n_q, are considered for the computation of the mean and variance of the quality characteristic. These additional design parameters are denoted by $n_{q,1}, n_{q,2}, ..., n_{q,p} \leq n_q$ and, in conjunction with the above, have also two possible values, a relaxed $\{n_{1,1}, n_{1,2}, ..., n_{1,p}\} \leq n_1$ and a tightened one $\{n_{2,1}, n_{2,2}, ..., n_{2,p}\} \leq n_2$.

The subgroup statistics $T_q^2 = n_q \cdot \left(\bar{x}' - \mu_0\right)^{\mathrm{T}} \cdot \Sigma^{-1} \cdot \left(\bar{x}' - \mu_0\right)$ and $\mathrm{E}_{q,t} = \sqrt{\frac{2 \cdot \left(n_q - 1\right)}{p}} \cdot \sum_{\rho=1}^{p} \ln\left(\frac{R_\rho}{\sigma_{0,\rho}}\right)$ are plotted on the two charts that monitor the mean vector and the covariance matrix, respectively and their values are compared to the warning and control limits of each chart.

The process is considered to be *IC* if both statistics are below the respective upper warning lines (central zone) ($T_q^2 \leq w_{mv}$ and $z_{E_q} \leq w_{cm}$). Then, the decision made at the *t*-th sampling inspection is that the process should continue its operation and relaxed parameters should be used for the next sampling $\left(a_t = 0\right)$. In case at least one of the two statistics lies between the upper warning and control lines (warning zone), but none of them outreaches the control limit, ($w_{mv} < T_q^2 \leq k_{mv,q}$ and $z_{E_q} \leq k_{cm,q}$ or $T_q^2 \leq k_{mv,q}$ and $w_{cm} < z_{E_q} \leq k_{cm,q}$), then the process should continue its operation, but the tightened set of parameters should be utilized for the next sampling $\left(a_t = 1\right)$. Finally, if at least one of the subgroup statistics outreaches the respective control limit (action zone) ($T_q^2 > k_{mv,q}$ and/or $z_{E_q} > k_{cm,q}$), then the process is halted, an inspection takes place and the process is perfectly restored to the *IC* state $\left(a_t = 2\right)$ if any assignable cause had indeed occurred.

The control policy for the two control charts is illustrated graphically in Figure 9-1.

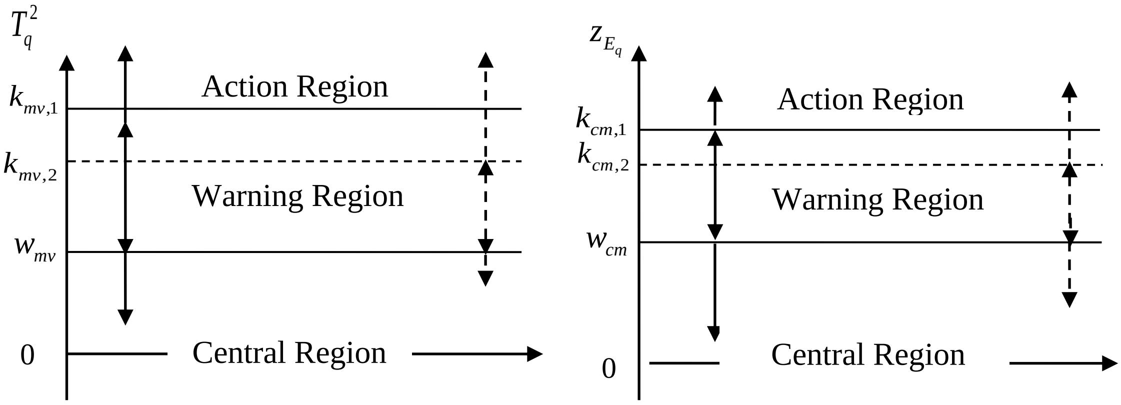

Figure 9-1: Regions of the VP_5 control scheme

Since there are $(m_{mv}+1)\times(r_{cm}+1)$ possible states for the mean vector and the covariance matrix and three possible decisions, as described above, the Markov chain has $(m_{mv}+1)\times(r_{cm}+1)\times 3$ possible states. Each state of the process is indicated by both the actual state of the process, denoted by (i,j) when the mean vector is affected by assignable cause *i* and the covariance matrix by assignable cause *j*, and the decision made, which is indicated by either $a_t=0$, if no action is taken, $a_t=1$, if the scheme warns for the effect of an assignable cause and $a_t=2$, if an alarm is issued. The transition probability matrix of the Markov chain is shown in Figure 9-2.

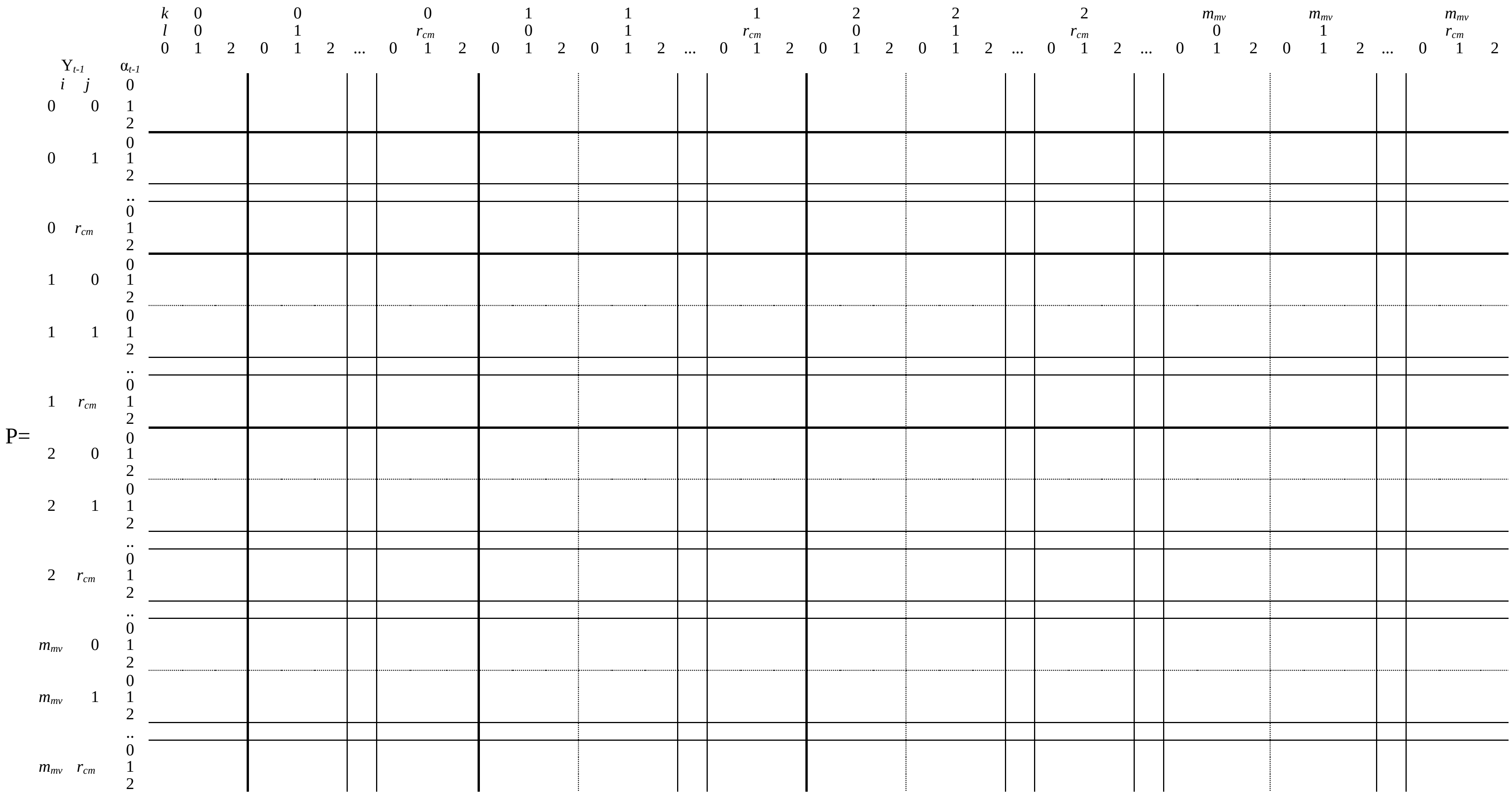

Figure 9-2: Transition Probability Matrix of the VP_5 control scheme

The probability of the process transition from state (i,j) to (k,l), $p_{\substack{(i,j)\\(k,l)}}(h_q)$, is the extension of equation (6.1) to the multivariate case and equals the product of multiplying the probability of the mean vector's transition to the respective transition of the covariance matrix:

$$p_{\substack{(i,j)\\(k,l)}}\left(h_q\right)=p_{\substack{mv,i\\k}}\left(h_q\right)\cdot p_{\substack{cm,j\\l}}\left(h_q\right)=\left(\begin{array}{l}\int_0^{h_q}\lambda_{mv(i\to k)}\cdot\exp\left(-v_{mv,i}\cdot t\right)\cdot\exp\left(-v_{mv,k}\cdot\left(h_q-t\right)\right)dt+\\+\int_0^{h_q}\sum_{y\in\{1,\ldots,m_{mv}\}\setminus\{k\}}\lambda_{mv(i\to y)}\cdot\exp\left(-v_{mv,i}\cdot t\right)\cdot p_{\substack{mv,y\\k}}\left(h_q-t\right)dt\end{array}\right)\cdot$$
$$\cdot\left(\begin{array}{l}\int_0^{h_q}\lambda_{cm(j\to l)}\cdot\exp\left(-v_{cm,j}\cdot t\right)\cdot\exp\left(-v_{cm,l}\cdot\left(h_q-t\right)\right)dt+\\+\int_0^{h_q}\sum_{z\in\{1,\ldots,r_{cm}\}\setminus\{l\}}\lambda_{cm(j\to z)}\cdot\exp\left(-v_{cm,j}\cdot t\right)\cdot p_{\substack{cm,z\\l}}\left(h_q-t\right)dt\end{array}\right)\qquad(9.5)$$

Now, let us see how the probability of the chart statistics to lie either in the central $(a_t=0)$, warning $(a_t=1)$, or action $(a_t=2)$ zone is derived.

Firstly, as already mentioned, the chart statistic T_q^2, follows a non-central chi-square distribution with *p* degrees of freedom and non-centrality parameter $n_q\cdot d_{(i,j)}^2$. By denoting the cumulative distribution function of T_q^2 by $F\left(x,p,n_q\cdot d_{(i,j)}^2\right)$, the probabilities for T_q^2 to be either in the central, warning or action zone, are:

$$P\left(T_q^2\le w_{mv}\right)=F\left(w_{mv},p,n_q\cdot d_{(i,j)}^2\right)\qquad(9.6)$$

$$P\left(w_{mv}<T_q^2\le k_{mv,q}\right)=F\left(k_{mv.q},p,n_q\cdot d_{(i,j)}^2\right)-F\left(w_{mv},p,n_q\cdot d_{(i,j)}^2\right)\qquad 9.7)$$

$$P\left(T_q^2>k_{mv,q}\right)=1-F\left(k_{mv,q},p,n_q\cdot d_{(i,j)}^2\right)\qquad(9.8)$$

Apparently, in case the mean vector equals its target value $(i=0)$ then, the statistic T_q^2 follows a chi-square distribution and the value of the non-centrality parameter equals zero $\left(n_q\cdot d_{(i,j)}^2=0\right)$.

On the other hand, as regards the normally distributed statistic E_q, an assignable cause *j* that affects the covariance matrix Σ, shifts only the mean of E_q from μ_{0,E_q} to μ_{j,E_q}, but its variance remains constant and equal to σ^2_{0,E_q}. This can easily be concluded from the general expressions of the mean and variance of E_q (equations (9.1) and (9.2)) and by taking into account that the occurrence of an assignable cause *j* shifts the standard deviation of each quality characteristic *ρ* from $\sigma_{0,\rho}$ to $\gamma_{j,\rho} \cdot \sigma_{0,\rho}$. Consequently, the occurrence of assignable cause *j* shifts only the value of the mean of E_q, upwards, to $\mu_{j,E_q} = \mu_{0,E_q} + \Delta_{j,q}$. The magnitude of the shift, denoted by $\Delta_{j,q}$, depends not only on the effect of assignable cause *j* to the standard deviation of each quality characteristic $\gamma_{j,\rho}$, but on the sample size n_q as well, and can be computed from the following equation:

$$\Delta_{j,\rho} = \sqrt{\frac{2 \cdot \left(n_q - 1\right)}{p}} \ln\left(\prod_{\rho=1}^{p} \gamma_{j,\rho}\right) \tag{9.9}$$

The respective probabilities for the standard normal variable z_{E_q} to lie either in the *central*, *warning* or *action* zone, under the effect of assignable cause *j*, are the following:

$$P\left(z_{E_q} \le w_{cm}\right) = \Phi\left(w_{cm} - \frac{\Delta_{j,q}}{\sigma_{0,E_q}}\right) \tag{9.10}$$

$$P\left(w_{cm} < z_{E_q} \le k_{cm,q}\right) = \Phi\left(k_{cm,q} - \frac{\Delta_{j,q}}{\sigma_{0,E_q}}\right) - \Phi\left(w_{cm} - \frac{\Delta_{j,q}}{\sigma_{0,E_q}}\right) \qquad 9.11)$$

$$P\left(z_{E_q} > k_{cm,q}\right) = 1 - \Phi\left(k_{cm,q} - \frac{\Delta_{j,q}}{\sigma_{0,E_q}}\right) \tag{9.12}$$

The probability of each possible decision can be now computed through equations (9.6)-(9.8) and (9.10)-(9.12):

$$P\left(a_t = 0\right) = P\left(T_q^2 \le w_{mv}\right) \cap P\left(z_{E_q} \le w_{cm}\right) \tag{9.13}$$

$$P\left(a_t=1\right)=\left(1-P\left(T_q^2>k_{mv,q}\right)\right)\cap\left(1-P\left(z_{E_q}>k_{cm,q}\right)\right)\setminus\left(P\left(T_q^2\leq w_{mv}\right)\cap P\left(z_{E_q}\leq w_{cm}\right)\right) \tag{9.14}$$

$$P\left(a_t=2\right)=P\left(T_q^2>k_{mv,q}\right)\cup P\left(z_{E_q}>k_{cm,q}\right) \tag{9.15}$$

As already mentioned, the transition probabilities equal the product of multiplying the probability of each possible transition to occur (equation (9.5)) to the probability of the decision made (equations (9.13)-(9.15)) and are derived from the following general expression:

$$Prob_{\substack{(i,j)a_{t-1}\\(k,l)a_t}}\left(h_q\right)=\begin{cases} p_{\substack{(i,j)\\(k,l)}}\left(h_q\right)\cdot F\left(w_{mv},p,n_q\cdot d_{(i,j)}^2\right)\cdot\Phi\left(w_{cm}-\dfrac{\Delta_{j,q}}{\sigma_{0,E_q}}\right) & a_t=0\\ p_{\substack{(i,j)\\(k,l)}}\left(h_q\right)\cdot\left[\begin{array}{l} F\left(k_{mv,q},p,n_q\cdot d_{(i,j)}^2\right)\cdot\Phi\left(k_{cm,q}-\dfrac{\Delta_{j,q}}{\sigma_{0,E_q}}\right)-\\ -F\left(w_{mv},p,n_q\cdot d_{(i,j)}^2\right)\cdot\Phi\left(w_{cm}-\dfrac{\Delta_{j,q}}{\sigma_{0,E_q}}\right)\end{array}\right] & a_t=1\\ p_{\substack{(i,j)\\(k,l)}}\left(h_q\right)\cdot\left[1-F\left(k_{mv,q},p,n_q\cdot d_{(i,j)}^2\right)\cdot\Phi\left(k_{cm,q}-\dfrac{\Delta_{j,q}}{\sigma_{0,E_q}}\right)\right] & a_t=2\end{cases} \tag{9.16}$$

Note that in case $a_{t-1}=2$, i.e., when the chart issues an alarm, the process always restarts its operation from the *IC* state and so the transition probabilities are computed from the following equation:

$$Prob_{\substack{(i,j)2\\(k,l)a_t}}\left(h_q\right)=Prob_{\substack{(0,0)0\\(k,l)a_t}}\left(h_1\right) \tag{9.17}$$

From the Markov chain theory, it is concluded that as the number of the collected samples *N* increases, the limiting probability for the process transition to Y_t and a_t is independent of the initial state of the process Y_{t-1} and a_{t-1} and is the unique non-negative solution of the following linear system:

$$\pi_{Y_t a_t}=\sum_{Y_{t-1}=(0,0)}^{(m_{mv},r_{cm})}\sum_{a_{t-1}=0}^{2}Prob_{\substack{Y_{t-1}a_{t-1}\\Y_t\ a_t}}\left(h_q\right)\cdot\pi_{Y_{t-1}a_{t-1}}\quad\text{and}\quad\sum_{Y_t=(0,0)}^{(m_{mv},r_{cm})}\sum_{a_t=0}^{2}\pi_{Y_t a_t}=1 \tag{9.18}$$

9.3 The Economic-Statistical Design

As it is already mentioned, the *OOC* operation cost per time unit, denoted by $K_{(i,j)}(h_q)$, when the process operates under the effect of state (i,j) at any given sampling instance, depends on the number of the assignable causes that may occur within an interval and their sequence of occurrence.

The computation of the aforementioned cost in presence of multiple assignable causes was described in Chapter 6 and its extension to the multivariate case is presented below:

$$K_{(i,j)}\left(h_q\right)=M_{(i,j)}\cdot h_q\cdot\exp\left(-\left(v_{mv,i}+v_{cm,j}\right)\cdot h_q\right)+$$
$$+\left(\begin{array}{l}\exp\left(-v_{cm,j}\cdot h_q\right)\cdot\int\limits_0^{h_q}\sum\limits_{y\in\{1,...,m_{mv}\}\setminus\{i\}}\left[\lambda_{mv(i\to y)}\cdot\exp\left(-v_{mv,i}\cdot t\right)\cdot\left(t\cdot M_{(i,j)}+K_{mv,(y,j)}\left(h_q-t\right)\right)\right]dt+\\+\exp\left(-v_{mv,i}\cdot h_q\right)\cdot\int\limits_0^{h_q}\sum\limits_{z\in\{1,...,r_{cm}\}\setminus\{j\}}\left[\lambda_{cm(j\to z)}\cdot\exp\left(-v_{cm,j}\cdot t\right)\cdot\left(t\cdot M_{(i,j)}+K_{cm,(i,z)}\left(h_q-t\right)\right)\right]dt\end{array}\right)+$$
$$+\sum_{y\in\{1,...,m_{mv}\}\setminus\{i\}}\ \sum_{z\in\{1,...,r_{cm}\}\setminus\{j\}}\left[\sum_{fn_1=1}^{m_{mv}-1}\sum_{fn_2=1}^{r_{cm}-1}ECK\left(h_q,\left(fn_1+fn_2\right)\right)_{\substack{(i,j)\\(y,z)}}\right]$$

(9.19)

The variables fn_1 and fn_2 denote the number of the assignable causes that occur within a transition step and affect the mean vector and the covariance matrix of the process, respectively, for a process transition from state (i,j) to (k,l).

Taking into account that the process is charged with different fixed and variable costs for sampling and testing for each quality characteristic, b_ρ and c_ρ $(\rho=1,2,...,p)$, respectively, and because different sample sizes are utilized from a collected sample of size n_q for the computation of the mean and variance of each quality characteristic $\left(n_{q,1},n_{q,2},...,n_{q,p}\right)\le n_q$, the average cost and duration of a transition step are computed from the following expressions:

$$EC = \sum_{\rho=1}^{p} b_\rho + \sum_{\rho=1}^{p}\sum_{k=0}^{m_{mv}}\sum_{l=0}^{r_{cm}} \pi_{(k,l)0} \cdot \left(c_\rho \cdot n_{1,\rho} + K_{(k,l)}\left(h_1\right)\right) +$$
$$+\sum_{\rho=1}^{p}\sum_{k=0}^{m_{mv}}\sum_{l=0}^{r_{cm}} \pi_{(k,l)1} \cdot \left(c_\rho \cdot n_{2,\rho} + K_{(k,l)}\left(h_2\right)\right) + \tag{9.20}$$
$$+\sum_{\rho=1}^{p}\sum_{k=0}^{m_{mv}}\sum_{l=0}^{r_{cm}} \pi_{(k,l)2} \cdot \left(c_\rho \cdot n_{1,\rho} + K_{(0,0)}\left(h_1\right) + L_{(k,l)}\right)$$

$$ET = h_1 \cdot \sum_{k=0}^{m_{mv}}\sum_{l=0}^{r_{cm}} \pi_{(k,l)0} + h_2 \cdot \sum_{k=0}^{m_{mv}}\sum_{l=0}^{r_{cm}} \pi_{(k,l)1} + \sum_{k=0}^{m_{mv}}\sum_{l=0}^{r_{cm}} \pi_{(k,l)2} \cdot \left(h_1 + T_{(k,l)} \right) \tag{9.21}$$

9.4 Optimization Problem

Let $DP_q = \left\{ n_q, h_q, w_{mv,q}, k_{mv,q}, w_{cm,q}, k_{cm,q} \right\}$ q=1,2 denote the optimal values of the design parameters and $N_q = \left\{ n_{q,1}, n_{q,2}, ..., n_{q,p} \right\}$, the optimal values of the individual sample sizes for each quality characteristic.

It should be noted that in accordance to the univariate case, without loss of generality, in order to reduce the complexity of the model and based on the conclusion of Park and Reynolds (1999) that the use of more than one warning limits leads to a relatively small cost reduction, we employed one warning limit for each control chart, i.e., $w_{mv,1} = w_{mv,2} = w_{mv}$ and $w_{cm,1} = w_{cm,2} = w_{cm}$.

The optimization problem can be formulated as follows:

$$\min_{DP_q, N_q} ECT$$
$$\text{s.t.} \quad h_1, h_2, n_1, n_2, w_{mv}, k_{mv,1}, k_{mv,2}, w_{cm}, k_{cm,1}, k_{cm,2} > 0$$
$$n_{q,1}, n_{q,2}, ..., n_{q,p} \le n_q$$
$$h_2 \le h_1 \tag{9.22}$$
$$n_2 \ge n_1$$
$$n_q > p$$
$$w_{mv} \le k_{mv,2} \le k_{mv,1}$$
$$w_{cm} \le k_{cm,2} \le k_{cm,1}$$
$$N_q \in \mathbb{Z}^+$$

The program software MATLAB (R2014a) is utilized for finding the optimal design parameters of the monitoring scheme DP_q^* and the sample sizes for each quality characteristic N_q^*.

9.5 An Illustrative Example

In this section, a real example from the maintenance division of a fighter aircraft squadron is utilized in order to illustrate the operation and evaluate the performance of the proposed scheme.

We consider the proposed scheme to be the main *SPC* tool for the on-line monitoring of the aircraft availability in a fighter aircraft squadron, which operates a fleet of 50 aircrafts. It should be noted that availability is defined as the proportion of the number of airworthy aircrafts out of the total number of aircrafts of the squadron and reflects the maintenance efficiency.

The early detection of a quality shift in the aircraft availability would be of great benefit to the squadron both in the short and long run. For example, it would provide greater breadth and depth of maintenance capability, save maintenance costs, increase the operational efficiency of the squadron, have a positive effect on the safety of operations etc.

Consequently, two quality characteristics $(p=2)$ are assumed to fully characterize the maintenance efficiency of the squadron: the daily Mean Time between Failure $(MTBF)$ and the daily Mean Time to Repair $(MTTR)$. Due to the dependence between failure and repair rates that often exists in the aircrafts, $MTBF$ and $MTTR$ are considered to be positive correlated $(r_c = 0.81)$.

From a statistical analysis of historical data and by taking into account the operational demands of the squadron, the target values of the means of the two quality characteristics are equal to $\mu_{0,MTBF} = 30$ hours and $\mu_{0,MTTR} = 4$ hours $\left(\mu_0' = (30,4)\right)$. On the other hand, the target values of the standard deviations are $\sigma_{0,MTBF} = 0.837$ and $\sigma_{0,MTTR} = 0.316$.

Three assignable causes affect the mean vector $(m_{mv} = 3)$ and three the covariance matrix $(r_{cm} = 3)$ of the monitored process. Specifically, bad lubricant control shifts only the mean of *MTBF* downwards to 29.75 hours $\mu_1' = (29.75, 4.0)$, poorly planned maintenance increases the mean of *MTTR* to 4.126 hours $\mu_2' = (30.0, 4.126)$ and poorly trained personnel affects both the means of *MTBF* and *MTTR* $\mu_3' = (29.582, 4.19)$. On the other hand, insufficient calibration processes increase the standard deviation of *MTBF* $(\gamma_{1,1} = 1.3, \gamma_{1,2} = 0)$, inappropriate handling procedures result in an upward shift of the standard deviation of *MTTR* $(\gamma_{2,1} = 0, \gamma_{2,2} = 1.4)$ and, finally, bad inventory management increases the standard deviations of both quality characteristics $(\gamma_{3,1} = 1.5, \gamma_{3,2} = 1.6)$.

Consequently, the covariance matrix is shifted by the three assignable causes from its target value $\Sigma_0 = \begin{bmatrix} 0.7 & 0.215 \\ 0.215 & 0.1 \end{bmatrix}$ to $\Sigma_1 = \begin{bmatrix} 1.183 & 0.2795 \\ 0.2795 & 0.1 \end{bmatrix}$ $(j = 1)$, $\Sigma_2 = \begin{bmatrix} 0.7 & 0.301 \\ 0.301 & 0.196 \end{bmatrix}$ $(j = 2)$ and $\Sigma_3 = \begin{bmatrix} 1.575 & 0.516 \\ 0.516 & 0.256 \end{bmatrix}$ $(j = 3)$, respectively.

For the specific example, three additional constraints should be implemented in the optimization problem (equation (9.22)): (a) because a reliable measurement of *MTBF* and *MTTR* can, only, be made at the end of a flight day, the sampling intervals should be integers ($h_1, h_2 \in \mathbb{Z}$), obviously positive; (b) $n_q \le h_q$, in order to assure that the production rate, i.e., flight days, will not constraint the measurement procedure; (c) as regards the statistical constraints, an upper bound has been set for *WARL* to be less or equal to one flight week, i.e., five flight days, $(WARL \le 5)$, in order to avoid a long-lasting out-of-control operation, which would result in a loss of confidence of the squadron to the proposed on-line monitoring tool.

Because the process continues its operation during search or repair the expressions of *EC* (equation (9.20)) would be slightly modified to account for this differentiation:

$$EC = \sum_{\rho=1}^{p} b_{\rho} + \sum_{\rho=1}^{p}\sum_{k=0}^{m_{mv}}\sum_{l=0}^{r_{cm}} \pi_{(k,l)0} \cdot \left(c_{\rho} \cdot n_{1,\rho} + K_{(k,l)}\left(h_1\right)\right) +$$

$$+\sum_{\rho=1}^{p}\sum_{k=0}^{m_{mv}}\sum_{l=0}^{r_{cm}} \pi_{(k,l)1} \cdot \left(c_{\rho} \cdot n_{2,\rho} + K_{(k,l)}\left(h_2\right)\right) + \tag{9.23}$$

$$+\sum_{\rho=1}^{p}\sum_{k=0}^{m}\sum_{l=0}^{r} \pi_{(k,l)2} \cdot \left(c_{\rho} \cdot n_{1,\rho} + K_{(0,0)}\left(h_1 + T_{(k,l)}\right) + L_{(k,l)}\right)$$

All the economic and statistical parameters of the process are presented in Table 9-2.

Table 9-2: Parameter set of the illustrative example for the VP_5 control scheme

Occurrence Rates (failures/day)	**Magnitude of Shifts**	**Costs** ($)			**Time Delays** (days)
		Sampling	Out-of-control Operation	Removal of assignable causes	
$\lambda_{mv(0\to1)} = \lambda_{cm(0\to1)} = 0.1$	$\delta_{1,1} = -0.3$				$T_{(0,0)} = 0.125$
$\lambda_{mv(0\to2)} = \lambda_{cm(0\to2)} = 0.1$	$\delta_{1,2} = \delta_{2,1} = 0$				$T_{(0,1)} = 0.25$
$\lambda_{mv(0\to3)} = \lambda_{cm(0\to3)} = 0.01$	$\delta_{2,2} = 0.4$	$c_1 = 10$	$M_{(0,1)} = 130$	$L_{(0,0)} = 150$	$T_{(0,2)} = 0.25$
$\lambda_{mv(1\to2)} = \lambda_{cm(1\to2)} = 0$	$\delta_{3,1} = -0.5$	$c_2 = 10$	$M_{(0,2)} = 150$	$L_{(0,1)} = 240$	$T_{(0,3)} = 0.35$
$\lambda_{mv(1\to3)} = \lambda_{cm(1\to3)} = 0.05$	$\delta_{3,2} = 0.6$	$b_1 = 5$	$M_{(0,3)} = 170$	$L_{(0,2)} = 240$	$T_{(1,0)} = 0.2$
$\lambda_{mv(2\to1)} = \lambda_{cm(2\to1)} = 0$	$\gamma_{1,1} = 1.3$	$b_2 = 5$	$M_{(1,0)} = 100$	$L_{(0,3)} = 260$	$T_{(1,1)} = 0.3$
$\lambda_{mv(2\to3)} = \lambda_{cm(2\to3)} = 0.05$	$\gamma_{1,2} = \gamma_{2,1} = 1$		$M_{(1,1)} = 140$	$L_{(1,0)} = 200$	$T_{(1,2)} = 0.4$
$\lambda_{mv(3\to1)} = \lambda_{cm(3\to1)} = 0$	$\gamma_{2,2} = 1.4$		$M_{(1,2)} = 170$	$L_{(1,1)} = 250$	$T_{(1,3)} = 0.45$
$\lambda_{mv(3\to2)} = \lambda_{cm(3\to2)} = 0$	$\gamma_{3,1} = 1.5$		$M_{(1,3)} = 210$	$L_{(1,2)} = 290$	$T_{(2,0)} = 0.2$
	$\gamma_{3,2} = 1.6$		$M_{(2,0)} = 120$	$L_{(2,0)} = 200$	$T_{(2,1)} = 0.4$
			$M_{(2,1)} = 150$	$L_{(2,1)} = 250$	$T_{(2,2)} = 0.5$
			$M_{(2,2)} = 180$	$L_{(2,2)} = 290$	$T_{(2,3)} = 0.55$
			$M_{(2,3)} = 210$	$L_{(2,3)} = 340$	$T_{(3,0)} = 0.35$
			$M_{(3,0)} = 150$	$L_{(3,0)} = 230$	$T_{(3,1)} = 0.45$
			$M_{(3,1)} = 160$	$L_{(3,1)} = 270$	$T_{(3,2)} = 0.55$
			$M_{(3,2)} = 200$	$L_{(3,2)} = 310$	$T_{(3,3)} = 0.6$
			$M_{(3,3)} = 250$	$L_{(1,3)} = 340$	
				$L_{(3,3)} = 350$	

The solution of the optimization problem dictates the optimal inspection policy to the maintenance officer who is responsible for the implementation of the proposed scheme. Specifically, the optimum values of the design parameters for our example are defined to be equal to: $h_1 = 11$, $h_2 = 10$, $n_1 = 3$, $n_2 = 6$, $w_{mv} = 0.1$, $k_{mv,1} = 24.4$, $k_{mv,2} = 24.2$, $w_{cm} = 0.1$, $k_{cm,1} = 3.3$, $k_{cm,2} = 2.5$.

As a result, a measurement of $MTBF$ and $MTTR$ should be done every 11 flight days $(h_1 = 11)$ for the last 3 $(n_1 = 3)$ flight days and the values of the two statistics, T_1^2 and z_{E_1}, should be compared to $k_{mv,1} = 24.4$ and $k_{cm,1} = 3.3$, respectively. In case, at least one of the chart statistic lies between the warning and control limit, but none of them outreaches the control limit, then, a new measurement should be made for the last 6 $(n_2 = 6)$ out of 10 $(h_2 = 10)$ flight days after the last inspection, and the "tightened" statistics T_2^2 and z_{E_2}, should be compared to $k_{mv,2} = 24.2$ and $k_{cm,2} = 2.5$, respectively. Finally, if the proposed scheme issues an alarm, a senior engineer should investigate and correct any abnormalities in the aircraft maintenance procedures. The optimal *ECT* equals 19.4235 $/ flight day and $WARL = 4.997$ flight days.

10. CONCLUSIONS AND FUTURE RESEARCH

10.1 Summary

In this thesis, fully adaptive control schemes are economically and statistically optimized in order to monitor processes under the assumption of multiple assignable causes that may affect both the location and scale of the process. For this purpose, the Markov chain theory has been utilized to model the operation of the proposed control schemes.

Specifically, a new adaptive control scheme for monitoring production processes subject to two assignable causes, one affecting the mean and the other the standard deviation of a quality characteristic, has been developed. The design parameters are allowed to vary in order to minimize the quality-related costs. The economic superiority of the proposed *VP* $\overline{X}$-*s* Shewhart control scheme, compared to less sophisticated control schemes has been verified through an extended numerical investigation.

Furthermore, a *VP* control chart for monitoring production processes subject to multiple assignable causes that affect the process mean of a quality characteristic has been developed. The design parameters are optimized economically with and without statistical performance constraints. The economic superiority of the proposed *VP* Shewhart control chart compared to less adaptive control charts is evaluated through an extended numerical investigation.

It should be mentioned that two different methods for the computation of the probability of a process transition and two methods for the computation of the *OOC* operation cost in the mathematically hard to formulate case of multiple assignable causes which may affect both the process mean and the process dispersion are developed. These approaches may be employed equivalently and their analytical presentation allows the reader to conceive the theoretical background of the modeling of this difficult problem.

Subsequently, a *VP* control scheme for monitoring production processes subject to a multiplicity of independent assignable causes that affect the process mean and/or the standard deviation of a specific quality characteristic is proposed. The values of the design parameters are derived through economic optimization but they also satisfy specific statistical constraints. The economic and statistical performance of the proposed control scheme is evaluated for an extended benchmark of scenarios. Finally, significant cost savings are verified by comparing the economic outcome of the scheme for each of the investigated cases, when instead of a multiple assignable cause mechanism, only one cause affecting the mean and another one the standard deviation are erroneously considered.

Additionally, a fully adaptive integrated maintenance and quality control scheme is developed. The proposed model can monitor processes subject to a multiple assignable cause mechanism which may affect both the process location and scale. Apart from the quality shifts that deteriorate the process performance, failures that cease the process operation may also occur. The detection of a quality shift from the control scheme triggers a *PM* action which is preferable to the *CM* action triggered in case of a failure. The values of the design parameters are selected so as to achieve economic optimization, assure acceptable statistical performance through specific statistical constraints and maximize the availability of the equipment. An extended benchmark of examples has been generated to compare the performance of the *VP* control scheme against simpler approaches.

Finally, a fully adaptive control scheme which tackles the common in a wide variety of industry applications problem of simultaneous monitoring of multiple correlated characteristics for processes whose location and scale may be affected by multiple independent assignable causes. This complicated problem is approached through a simple on-line monitoring tool which utilizes the commonly used Hotelling's statistic and the familiar to practitioners sample range in order to detect the occurrence of an/some assignable cause/es. The economic-statistical design of the control scheme is modeled as an optimization problem and the values of the optimal parameters are defined through an exhaustive algorithm. A realistic example from the aviation industry is employed to demonstrate the applicability and evaluate the performance of the proposed scheme.

It is apparent that each of the proposed models deals with real-world industry problems and can be implemented in a wide variety of processes, despite the fact that the general problem setting of the models results, unavoidably, in a hard mathematical formulation. However, through suitable software with a user-friendly, intuitive interface, the complexity of the mathematical model does not limit the applicability of the control scheme for practitioners.

Specifically, after a comprehensive initialization process, implemented by the quality division of the industry, in order to define the statistical and economic parameters of the monitored process (inputs), the software would dictate the optimal design parameters (output). Then, in the production line, the practitioners should only enter iteratively the measured values of the quality characteristic from each sample (inputs) in the program, which processes these data and defines the optimum sampling policy (output) each time.

10.2 Conclusions

The development of fully adaptive control schemes for monitoring processes subject to multiple assignable causes that affect both the process mean and the dispersion of the process led to the following useful conclusions.

First of all, the proposed *VP* control schemes have a better economic performance compared to respective control schemes with fewer adaptive parameters. This conclusion has been verified through multiple comparisons for a wide benchmark of different cases and for the majority of the developed models.

Moreover, the tightened value of the sampling interval (h_2) equals the minimum allowable value in the majority of the examined cases. Regardless of whether the value of h_2 is allowed to be equal to zero, as examined in Chapter 4, or not, a warning of the control schemes dictates an immediate sampling. This fact leads to the conclusion that the *DS* policy is the most economically advantageous sampling strategy in a wide variety of scenarios.

Another conclusion is that an erroneous consideration of a single assignable cause mechanism when multiple quality shifts are possible to occur, has significant economic repercussions on the mean total quality-related costs. It is apparent that this conclusion enhances the necessity of application of the proposed models in processes subject to multiple quality shifts.

10.3 Future Research

Interesting topics for future research may include the development of regression models based on the results of the numerical analysis sections, in order to figure out the effect of each parameter to the economic performance of the proposed control schemes.

A sensitivity analysis of the process parameters misestimation to evaluate its effect to long-run average cost per time-unit may be considered. Moreover, the evaluation of the robustness of the proposed control scheme to the assumption of normally distributed quality characteristics may be useful.

An area for potential future work is to remove the assumption of normally distributed quality characteristics and develop non-parametric univariate and multivariate control schemes for monitoring processes subject to multiple assignable causes affecting location and/or scale.

Furthermore, the general conclusion of the performance superiority of adaptive control schemes against the respective, less-adaptive ones, makes the allowance of more than two sets of design parameters, and the evaluation of how the economic performance is affected, an interesting topic for future work.

Finally, as regards the proposed multivariate control scheme, i.e., VP_5, a comparison with other approaches could be considered. In addition, the extension of the proposed model to also monitor changes in the correlation structure and the extension to multivariate processes where autocorrelation within the variables is also present, may be considered.

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APPENDICES

APPENDIX A Nomenclature

$\Gamma(\cdot)$	probability density function of gamma distribution
$\Phi(\cdot)$	cumulative density function of the standard normal distribution
$\psi(\cdot)$	probability density function of digamma distribution
$\psi_1(\cdot)$	probability density function of trigamma distribution
$F(\cdot)$	cumulative density function of noncentral chi-square distribution
$c_{4,q}$	coefficient for the sample size: $q=1$ $(q=2)$ when relaxed (tightened) parameters are used
X	critical to quality characteristic
X_m	critical to quality multivariate variable
μ_T	target mean
σ_T	target standard deviation
$\mu_{0,\rho}$	in-control mean of ρ^{th} quality characteristic (μ_0 in case $p=1$)
$\sigma_{0,\rho}$	in-control standard deviation of ρ^{th} quality characteristic (σ_0 in case $p=1$)
$\mu_0{}'$	in-control mean vector
Σ_0	in-control covariance matrix
$\mu_{i,\rho}$	out-of-control process mean of ρ^{th} quality characteristic when assignable cause *i* has occurred $\left(\mu_{i,\rho}=\mu_{0,\rho}+\delta_{i,\rho}\cdot\sigma_{0,\rho}\right)$ (μ_i in case $p=1$)
$\sigma_{j,\rho}$	out-of-control standard deviation of ρ^{th} quality characteristic when assignable cause *j* has occurred $\left(\sigma_{j,\rho}=\gamma_{j,\rho}\cdot\sigma_{0,\rho}\right)$ (σ_j in case $p=1$)

μ_i' out-of-control mean vector when assignable cause *i* has occurred $\mu_i' = \left(\mu_1 + \delta_{i,1} \cdot \sigma_1, ..., \mu_p + \delta_{i,p} \cdot \sigma_p\right)$

Σ_j out-of-control covariance matrix when assignable cause *j* has occurred

μ_{0,E_q} in-control of statistic E_q

μ_{i,E_q} out-of-control mean of statistic E_q when assignable cause *i* has occurred

σ_{0,E_q} in-control standard deviation of statistic E_q

σ_{j,E_q} out-of-control standard deviation of statistic E_q when assignable cause *j* has occurred

D_j diagonal matrix of the standard deviation vector when assignable cause *j* has occurred $\left(D_j = diag\left(\sigma_{j,\rho}\right)\right)$

$\delta_{i,\rho}$ magnitude of the shift in the process mean of ρ^{th} quality characteristic due to assignable cause *i* (δ_i in case $p=1$)

$\gamma_{j,\rho}$ magnitude of the shift in the standard deviation of ρ^{th} quality characteristic due to assignable cause *j* (γ_j in case $p=1$)

$\Delta_{j,q}$ magnitude of the shift in the mean of statistic E_t when a sample of size n_q is collected $\left(\Delta_{j,q} = \sqrt{\frac{2\left(n_q - 1\right)}{p}} \cdot \ln\left(\prod_{\rho=1}^{p} \gamma_{j,\rho}\right)\right)$

$d_{(i,j)}$ Mahalabonis distance of a sample mean vector from the in-control process mean vector for operation under the effect of state (i, j)

m total number of different assignable causes that may affect the process mean

r total number of different assignable causes that may affect the standard deviation of the process

m_{mv} total number of different assignable causes that may affect the mean vector

r_{cm} total number of different assignable causes that may affect the covariance matrix

p total number of correlated quality characteristics

r_c correlation coefficient

z_t standardized sample mean at t^{th} sampling instance

s_t standard deviation of a sample at t^{th} sampling instance

$E_{q,t}$ sample statistic for estimation of multivariate process dispersion at t^{th} sampling instance for relaxed $(q=1)$ or tightened sampling $(q=2)$

$z_{E_{q,t}}$ standardized value of $E_{q,t}$ at t^{th} sampling instance for relaxed $(q=1)$ or tightened sampling $(q=2)$

$T^2_{q,t}$ Hotelling's statistic at t^{th} sampling instance for relaxed $(q=1)$ or tightened sampling $(q=2)$

R_ρ range of ρ^{th} quality characteristic of a sample of size n_q $\left(R_\rho = \max\left[x_{1,\rho},...,x_{n_q,\rho}\right] - \min\left[x_{1,\rho},...,x_{n_q,\rho}\right]\right)$

$\bar{x}'$ sample mean vector

$\lambda_{x(i\to k)}$ occurrence rate of assignable cause k when the process mean is already under the effect of assignable cause i

$\lambda_{s(j\to l)}$ occurrence rate of assignable cause l when the standard deviation of the process is already under the effect of assignable cause j

$\lambda_{Fx|i}$ failure rate when the process mean is under the effect of assignable cause i

$\lambda_{Fs|j}$ failure rate when the standard deviation of the process is under the effect of assignable cause j

$\lambda_{mv(i\to k)}$ occurrence rate of assignable cause k when the process mean vector is already under the effect of assignable cause i

$\lambda_{cm(j \to l)}$ occurrence rate of assignable cause l when the covariance matrix of the process is already under the effect of assignable cause j

$v_{x,i}$ transition rate to any inferior state of the process mean when the mean is already under the effect of assignable cause i

$v_{s,j}$ transition rate to any inferior state of the standard deviation of the process when the standard deviation is already under the effect of assignable cause j

$v_{mv,i}$ transition rate to any inferior state of the process mean vector when the mean vector is already under the effect of assignable cause i

$v_{cm,j}$ transition rate to any inferior state of the covariance matrix of the process when the covariance matrix is already under the effect of assignable cause j

$n_1, h_1, w_x, k_{x,1}, w_s, k_{s,1}$ univariate control scheme's parameters for relaxed sampling

$n_2, h_2, w_x, k_{x,2}, w_s, k_{s,2}$ univariate control scheme's parameters for tightened sampling

$n_1, h_1, w_{mv}, k_{mv,1}, w_{cm}, k_{cm,1}$ multivariate control scheme's parameters for relaxed sampling

$n_2, h_2, w_{mv}, k_{mv,2}, w_{cm}, k_{cm,2}$ multivariate control scheme's parameters for tightened sampling

$n_{1,\rho}$ relaxed sample size of ρ^{th} quality characteristic

$n_{2,\rho}$ tightened sample size of ρ^{th} quality characteristic

$UWL_{s,q}$ upper warning limit of the *s* control chart

$UCL_{s,q}$ upper control limit of the *s* control chart

Y_t current state of the process at *t*-th inspection

a_t decision that should be taken according to the last point plotted on the control chart at *t*-th inspection

$q_{\substack{(i,j)\\(u,v)}}$ restoration probability of the process from state (i, j) to a superior state (u,v)

$p_{\substack{(i,j)\\(k,l)}}\left(h_q\right)$ probability for the process moving from state $\left(i,j\right)$ to state $\left(k,l\right)$ during a sampling interval of duration h_q

$p_{x,\substack{i\\k}}\left(h_q\right)$ probability for the process mean moving from state *i* to state *k* during a sampling interval of duration h_q

$p_{s,\substack{j\\l}}\left(h_q\right)$ probability for the process standard deviation moving from state *j* to state *l* during a sampling interval of duration h_q

$p_{mv,\substack{i\\k}}\left(h_q\right)$ probability for the process mean vector moving from state *i* to state *k* during a sampling interval of duration h_q

$p_{cm,\substack{j\\l}}\left(h_q\right)$ probability for the process covariance matrix moving from state *j* to state *l* during a sampling interval of duration h_q

$Prob_{\substack{Y_{t-1}a_{t-1}\\Y_t a_t}}\left(h_q\right)$ transition probabilities during a sampling interval of duration h_q

$\pi_{Y_t a_t}$ steady-state probabilities (in case of failure $\left(Y_t = F\right)$ $\pi_{Y_t a_t}$ is denoted by π_F)

c_ρ variable sampling cost per unit of ρ^{th} quality characteristic (c in case $p=1$)

b_ρ fixed sampling cost of ρ^{th} quality characteristic (b in case $p=1$)

$L_{(i,j)}$ cost of process restoration from state $\left(i,j\right)$ ($L_{(0,0)}$ cost of a false alarm)

L_F cost of a *CM* action

$M_{(i,j)}$ cost per time unit for operation under the effect of state $\left(i,j\right)$

P transition probability matrix

P_0 correlation matrix

$T_{(i,j)}$ time delay for a process restoration from state $\left(i,j\right)$ ($T_{(0,0)}$ time delay to detect a false alarm)

T_F time delay for a *CM* action

fn_1 number of assignable causes that occur within a transition step and affect the process mean

fn_2 number of assignable causes that occur within a transition step and affect the standard deviation of the process

$SC_{\substack{(i,j)\\(k,l)}}$ set of all the possible scenarios for a process transition from state (i, j) to (k,l)

$\Pr\left(h_q,(fn_1+fn_2)\right)_{\substack{(i,j)\\(k,l)}}$ probability of a specific scenario to occur in order for the process to move from state (i, j) to (k,l) when (fn_1+fn_2) assignable causes occur within the interval of duration h_q

$CK\left(h_q,(fn_1+fn_2)\right)_{\substack{(i,j)\\(k,l)}}$ out-of-control operation cost of a specific scenario to occur in order for the process to move from state (i, j) to (k,l) when (fn_1+fn_2) assignable causes occur within the interval of duration h_q

$ECK\left(h_q,(fn_1+fn_2)\right)_{\substack{(i,j)\\(k,l)}}$ expected out-of-control operation cost in order for the process to move from state (i, j) to (k,l) when (fn_1+fn_2) assignable causes occur within the interval of duration h_q

$\tau^{(g)}\left(h_q\right)$ expected time of the occurrence of the g^{th} assignable cause of a specific scenario within an interval of duration h_q

$\tau_{IC/(0,0)}\left(h_q\right)$ expected *IC* time of the process within a sampling interval of duration h_q

$\tau_{OOC/(i,j)}\left(h_q\right)$ expected *OOC* time of the process for state (i, j) being the initial state of a sampling interval of duration h_q

$K_{(i,j)}\left(h_q\right)$ expected *OOC* operation cost of the process, for state (i, j) being the initial state of a sampling interval of duration h_q

$K_{x,(i,j)}\left(h_q\right)$ expected *OOC* operation cost of the process, for state $\left(i,j\right)$ being the initial state of a sampling interval of duration h_q in cases where the standard deviation of the process remains unaffected during the interval

$K_{s,(i,j)}\left(h_q\right)$ expected *OOC* operation cost of the process, for state $\left(i,j\right)$ being the initial state of a sampling interval of duration h_q in cases where the process mean remains unaffected during the interval

$K_{mv,(i,j)}\left(h_q\right)$ expected *OOC* operation cost of the process, for state $\left(i,j\right)$ being the initial state of a sampling interval of duration h_q in cases where the covariance matrix of the process remains unaffected during the interval

$K_{cm,(i,j)}\left(h_q\right)$ expected *OOC* operation cost of the process, for state $\left(i,j\right)$ being the initial state of a sampling interval of duration h_q in cases where the process mean vector remains unaffected during the interval

ARL_0 in-control average run length

ANOF average number of false alarms per time unit

ATC average time of a cycle

α *Type I* error probability

$1-\beta$ control chart's power

EATR expected time from the occurrence until the detection and removal of an assignable cause

EATS expected average time to signal

WARL weighted out-of-control average run length

EC expected cost of a transition step, i.e., the actual duration between the beginning of two successive sampling intervals

ET expected duration of a transition step, i.e., the actual duration between the beginning of two successive sampling intervals

ECT expected cost per time unit $\left(=EC/ET\right)$

AT	expected uptime of equipment
EA	expected long-run availability $\left(= AT / ET\right)$